On Integrality of SO(n)-Level 2 TQFTs

Dissertation

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Abstract

In this thesis, we study properties of the TQFT associated to the modular category $\mathrm{SO}(p)_2$ for an odd prime p. We compute the associated representation $\rho_{1,1}^a : \widetilde{\Gamma_{1,1}} \to \mathrm{GL}(\mathbb{V}_{1,1}^a)$ of the central extension of the mapping class group $\widetilde{\Gamma_{1,1}}$ of the surface of genus one with one boundary component specialized at a simple object a of $\mathrm{SO}(p)_2$. Let $\zeta = \exp\left(\frac{2}{p}\pi i\right)$ be a primitive p-th root of unity. We show that for each a, there exists a full-rank free lattice Λ_a over the ring of integers $\mathbb{Z}[\zeta, i]$ inside the specialized space $\mathbb{V}_{1,1}^a$ that is preserved by the $\widetilde{\Gamma_{1,1}}$ action. We also show that for each a, the image of $\rho_{1,1}^a$ is a finite subgroup of $\mathrm{GL}(\mathbb{V}_{1,1}^a)$. Finally, we relate the representations $\mathrm{GL}(\mathbb{V}_{1,1}^a)$ to the Weil representation over finite fields.

To my parents and my wife

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Chapter 1

Introduction

Il faut imaginer Sisyphe heureux.

Albert Camus, *Le mythe de Sisyphe*

The formalism of Topological Quantum Field Theory (TQFT) was first introduced by Atiyah [Ati88] and Witten [Wit89], and rigorous constructions of TQFTs using quantum groups were invented by Reshetikhin-Turaev around 1990 [RT91]. Any TQFT specializes, in particular, to invariants of knots, links, and closed 3-manifolds. For example, the TQFTs arising from the quantum groups at roots of unity associated to the simple Lie algebra $\mathfrak{sl}_2$ are closely related to the celebrated Jones polynomial. Besides the quantum invariants, TQFTs also provide a series of (projective) representations of the mapping class groups of compact oriented surfaces, which are powerful tools to study these groups.

The concept of TQFTs evolved over the years [Tur10, KL01]. In [RT91, KL01], TQFTs are constructed from the so-called modular categories. A modular category is a finite, semisimple, abelian, $\mathbb{C}$-linear, rigid, monoidal, ribbon category such that the braiding is non-degenerate, see [EGNO15]. In this thesis, a TQFT is a monoidal functor

$$\mathcal{V} : \boldsymbol{Cob}^{\bullet} \longrightarrow \mathcal{C} \qquad (1.0.1)$$

from the category of 2-framed, relative cobordism category of compact oriented connected surfaces with one boundary component to a modular category $\mathcal{C}$.

A systematic way to construct modular categories is to use the representation theory of quantum groups at roots of unity, see for example [APW95, And92, RT91]. The TQFT representations of mapping class groups arising from such modular categories are finite-dimensional and can be defined over a cyclotomic field $\mathbb{Q}(\zeta)$ where ζ is a root of unity. In particular, the corresponding quantum invariants are elements of $\mathbb{Q}(\zeta)$.

Algebraic properties of quantum invariants given by quantum groups are extensively studied since the integrality result of of Murakami [Mur94, Mur95], who showed that the SU(2)- and SO(3)-invariants of 3-manifolds are algebraic integers. More precisely, those invariants are elements of the ring of integers of $\mathbb{Q}(\zeta, i)$, namely, $\mathbb{Z}[\zeta, i]$, where ζ is a root of unity of prime order. The result was reproved in [MR97], generalized to all classical Lie types in [MW98, TY99], then to all Lie types by Le [Le03]. These results helped us relate the quantum invariants to other invariants such as the Casson invariant [KM91, Mur94, Mur95] and the Ohtsuki series [Oht95, Oht96, Le03].

A natural question to ask is whether one can define the whole quantum group TQFT over $\mathbb{Z}[\zeta]$ for some root of unity ζ (in the sense of [Tur10, Definition II.1.4]). Or, if one is more topologically inclined, he or she may ask if the representations of the mapping class groups given by a quantum group TQFT are defined over $\mathbb{Z}[\zeta]$. If either of these questions admits a positive answer, we can use the ideal structure of $\mathbb{Z}[\zeta]$ to detect refined topological information or properties. For example, in [GM07] the authors studied the Frohman-Kania-Bartoszynska ideal invariants [FKB01] using the integral SO(3)-TQFT. The integrality of the SO(3)-TQFT, especially, the explicit $\mathbb{Z}[\zeta]$-modules on which the mapping class groups act, enables the authors to give an explicit finite set of generators for the Frohman-Kania-Bartoszynska ideal. Using the generators of the Frohman-Kania-Bartoszynska ideal, the authors were able to exhibit a family of examples of 3-manifolds with the homology of a solid torus which cannot embed in S^3. Another possible application of integral TQFTs is that we can reduce these TQFTs by the natural reduction map $\mathbb{Z}[\zeta] \to \mathbb{Z}/p\mathbb{Z}[y]/(y^{p-1})$ to get the so-called p-modular TQFTs. The p-modular TQFTs have rich connections to topological information of 3-manifolds, such as the Casson-Lescop invariants and the Milnor torsion, see for example, [Ker02]. We may also answer questions such as the finiteness of the images of these representations with the help of the integrality results.

For the SO(3)-TQFTs associated to an odd prime p (also referred to as the V_p-theory), Gilmer, Masbaum and van Wamelen first constructed integral bases for genus one and two [GMvW04]. Then Gilmer and Masbaum generalized the results to arbitrary genus in [GM07], hence completed the construction of the integral SO(3)-TQFT. Note that in [GMvW04] and [GM07], the TQFTs under consideration are those associated to closed surfaces, and the authors restricted the type of cobordisms to the so-called targeted cobordisms.

In this thesis, we study the integrality properties of the TQFTs associated to a family of modular categories associated to the quantum groups $U_{q'}(\mathfrak{so}(p))$, where p is an odd prime, and $q' = \exp\left(\frac{1}{2p}\pi i\right)$. We denote these modular categories by $\mathrm{SO}(p)_2$. The $\mathrm{SO}(p)_2$ categories are interesting subjects to study. The square of the quantum dimensions of the simple objects in $\mathrm{SO}(p)_2$ are integers, and there exists a subcategory that has the same fusion rules as $Rep(D_{2p})$, the category of finite dimensional complex representations of the Dihedral group D_{2p}. The relationship between these two categories are worth studying. The category $\mathrm{SO}(p)_2$ also appears as the equivariantization [DGNO10] of the Tamabara-Yamagami categories [TY98]. Moreover, $\mathrm{SO}(p)_2$ is intimately related to the metaplectic link invariants of Goldschmidt-Jones

[GJ89]. Using the relationship between $\mathrm{SO}(p)_2$ and the metaplectic link invariant construction, Rowell and Wenzl proved that the braid group representations arising from the TQFTs associated to $\mathrm{SO}(p)_2$ have finite image [RW17]. Their result provides evidences of the the *Property-F conjecture* [NR11], which posts that the braid group representations arising from a TQFT associated to a modular category $\mathcal{C}$ have finite image if and only if the square of the quantum dimensions of all of the simple objects in $\mathcal{C}$ are integers.

From a physics point of view, the $\mathrm{SO}(p)_2$-TQFTs emerge as an algebraic tool to organize phases of matter in the context of topological quantum computing [HNW14].

Let $\Sigma_{g,1}$ be a specific connected compact oriented surface of genus g with one boundary component. Let $\Gamma_{g,1}$ be the mapping class group of $\Sigma_{g,1}$, which is the group of isotopy classes of orientation preserving homeomorphisms of $\Sigma_{g,1}$ onto itself such that the boundary component $\partial(\Sigma_{g,1})$ is fixed pointwise. Let $\widetilde{\Gamma_{g,1}}$ be a central extension of $\Gamma_{g,1}$ by $\mathbb{Z}$, carrying the 2-framing information.

Let p be an odd prime. Denote by $\mathrm{Irr}\,(\mathrm{SO}(p)_2)$ the set of representatives of isomorphism classes of simple objects in $\mathrm{SO}(p)_2$. Let $\mathcal{V} : \boldsymbol{Cob}^\bullet \to \mathrm{SO}(p)_2$ be the TQFT associated to $\mathrm{SO}(p)_2$ and let

$$\mathcal{V}^a : \boldsymbol{Cob}^\bullet \xrightarrow{\mathcal{V}} \mathrm{SO}(p)_2 \xrightarrow{\mathrm{Hom}_{\mathrm{SO}(p)_2}(a,\cdot)} \boldsymbol{Vec}, \tag{1.0.2}$$

be the specialization of $\mathcal{V}$ to a simple object $a \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$. Let $\mathbb{V}^a_{g,1} = \mathcal{V}^a(\Sigma_{g,1})$ be the specialized space associated to $\Sigma_{g,1}$. The TQFT axioms (see [KL01]) imply that for any simple object a, we have a finite dimensional linear representation of the central extension of the mapping class group

$$\rho^a_{g,1} : \widetilde{\Gamma_{g,1}} \longrightarrow \mathrm{GL}(\mathbb{V}^a_{g,1}). \tag{1.0.3}$$

In this thesis, we focus on the representation $\rho_{1,1}^a$ of $\widetilde{\Gamma_{1,1}}$. Let $\zeta := \exp\left(\frac{2}{p}\pi i\right)$ be a primitive p-th root of unity. The main result of this thesis is given in the following theorem.

Theorem 1.0.1 (Main Theorem)**.** *For any simple object $a \in \operatorname{Irr}(\mathrm{SO}(p)_2)$, there exists a $\widetilde{\Gamma_{1,1}}$-invariant, full-rank, free lattice $\Lambda_a \subset \mathbb{V}_{1,1}^a$ over the ring $\mathbb{Z}[\zeta, i]$.*

The Main Theorem will follow from Theorem 5.1.1, Theorem 5.2.1, and Theorem 5.3.1.

With the help of the integral bases (bases of the lattice Λ_a) and [NS10, Theorem 7.1], we are able to show the following finiteness result.

Theorem 1.0.2. *For any simple object $a \in \operatorname{Irr}(\mathrm{SO}(p)_2)$, $\rho_{1,1}^a(\widetilde{\Gamma_{1,1}})$ is a finite subgroup of $\mathrm{GL}(\mathbb{V}_{1,1}^a)$.*

In addition, we can also relate the integral representations obtained from the Main Theorem to the Weil representations over finite fields and the Birman-Craggs-Johnson homomorphisms.

The structure of the thesis is as follows: in Chapter 2 we give the definition of modular categories. We also introduce a graphical calculus to perform calculations with morphisms of a modular category. This will be an important tool to compute $\rho_{1,1}^a$ and to construct integral bases. In Chapter 3, we introduce the notion of a TQFT and the corresponding $\widetilde{\Gamma_{g,1}}$ representations $\rho_{g,1}^a$. We give an algorithm to compute $\rho_{g,1}^a$ using the tangle presentation of $\widetilde{\Gamma_{g,1}}$, and compute $\rho_{1,1}^a$ as an example. In Chapter 4 we introduce the $\mathrm{SO}(p)_2$ categories, together with some notations for future references. In Chapter 5 we use the techniques introduced in the previous chapters to prove our main integrality result. Finally, in Chapter 6, we give examples of explicit expressions of $\rho_{1,1}^a$ evaluated on generators of $\widetilde{\Gamma_{1,1}}$ with respect to the integral bases and their reductions mod p. We will also discuss the image finiteness and the relationship between $\rho^a_{1,1}$ and the Weil representations over finite fields.

Chapter 2

Modular Categories

2.1 Braided rigid monoidal categories

We assume that the reader is familiar with the basic notions of categories, functors and natural transformations.

2.1.1 Monoidal categories

The notion of monoidal categories dates back to Bénabou [Bén63] and Mac Lane [ML63] as a formalism to study "categories with multiplications". We follow [ML98] to give a precise definition.

Definition 2.1.1. *A* monoidal category *is a tuple* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$*, where*

$\mathcal{C}$ *is a category,*

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *is a (covariant) bifunctor,*

$\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$ *is an object, and*

$\alpha : \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes) \dot{\to} \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}})$*,* $\lambda : \otimes \circ (\mathbb{1} \times \mathrm{id}_{\mathcal{C}}) \dot{\to} \mathrm{id}_{\mathcal{C}}$*,* $\rho : \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \mathbb{1}) \dot{\to} \mathrm{id}_{\mathcal{C}}$ *are natural isomorphisms. In other words,*

$$\alpha_{U,V,W} : U \otimes (V \otimes W) \xrightarrow{\cong} (U \otimes V) \otimes W, \tag{2.1.1}$$

$$\lambda_V : \mathbb{1} \otimes V \xrightarrow{\cong} V \tag{2.1.2}$$

and

$$\rho_V : V \otimes \mathbb{1} \xrightarrow{\cong} V \tag{2.1.3}$$

are natural in any $U, V, W \in \mathrm{Ob}(\mathcal{C})$*. We call* α *the* associativity constraint*, and we call* λ *(*ρ*) the* left (right) unit constraint *respectively.*

The tuple has to satisfy the following axioms.

Pentagon axiom*:*

$$\begin{array}{ccc} & U \otimes (V \otimes (W \otimes X)) & \\ {}^{\alpha_{U,V,W\otimes X}}\swarrow & & \searrow^{\mathrm{id}_U \otimes \alpha_{V,W,X}} \\ U \otimes ((V \otimes W) \otimes X) & & (U \otimes V) \otimes (W \otimes X) \\ \downarrow{\scriptstyle \alpha_{U,V\otimes W,X}} & & \downarrow{\scriptstyle \alpha_{U\otimes V,W,X}} \\ (U \otimes (V \otimes W)) \otimes X & \xrightarrow{\alpha_{U,V,W} \otimes \mathrm{id}_X} & ((U \otimes V) \otimes W) \otimes X \end{array} \tag{2.1.4}$$

commutes for any $U, V, W, X \in \mathrm{Ob}(\mathcal{C})$*.*

Triangle axiom*:*

$$\begin{array}{ccc} V \otimes (\mathbb{1} \otimes W) & \xrightarrow{\alpha_{V,\mathbb{1},W}} & (V \otimes \mathbb{1}) \otimes W \\ & {}_{\mathrm{id}_V \otimes \lambda_W}\searrow \quad \swarrow_{\rho_V \otimes \mathrm{id}_W} & \\ & V \otimes W & \end{array} \tag{2.1.5}$$

commutes for any $V, W \in \mathrm{Ob}(\mathcal{C})$*.*

For simplicity we will denote a monoidal category by $\mathcal{C}$.

Remark 2.1.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ be a monoidal category. Let $V_1, ..., V_n \in \mathrm{Ob}(\mathcal{C})$. Mac Lane's Coherence Theorem [ML98, Section VII.2] implies that any two parenthesized tensor products of $V_1, ..., V_n$ in this order with arbitrary intersections of $\mathbb{1}$ are isomorphic by a unique isomorphism obtained by composing α, ρ and λ and their inverses (possibly tensored with identity morphisms). Therefore, in the following context, we are free to drop parentheses and suppress α, λ, ρ and their inverses.

Now we need a suitable notion of functors between monoidal categories that "preserve the monoidal structures". This inspires the following definition.

Definition 2.1.2. *A* monoidal functor *between two monoidal categories* $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ *and* $(\mathcal{D}, \otimes', \mathbb{1}', \alpha', \lambda', \rho')$ *is a triple* (F, ϕ_0, ϕ_2)*, where*

$F : \mathcal{C} \to \mathcal{D}$ *is a functor,*

$\phi_0 : F(\mathbb{1}) \to \mathbb{1}'$ *is an isomorphism,*

$\phi_2 : \otimes' \circ (F \times F) \dot{\to} F \circ \otimes$ *is a natural isomorphism. In other words,*

$$\phi_2(V, W) : F(V) \otimes' F(W) \xrightarrow{\cong} F(V \otimes W) \tag{2.1.6}$$

is natural in any $V, W \in \mathrm{Ob}(\mathcal{C})$*.*

The conditions that the triple has to satisfy is that the diagrams

$$\begin{array}{ccc} F(U) \otimes' (F(V) \otimes' F(W)) & \xrightarrow{\alpha'_{F(U),F(V),F(W)}} & (F(U) \otimes' F(V)) \otimes' F(W) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{F(U)} \otimes' \phi_2(V,W)} & & \Big\downarrow{\scriptstyle \phi_2(U,V) \otimes' \mathrm{id}_{F(W)}} \\ F(U) \otimes' F(V \otimes W) & & F(U \otimes V) \otimes' F(W) \\ \Big\downarrow{\scriptstyle \phi_2(U, V \otimes W)} & & \Big\downarrow{\scriptstyle \phi_2(U \otimes V, W)} \\ F(U \otimes (V \otimes W)) & \xrightarrow{F(\alpha_{U,V,W})} & F((U \otimes V) \otimes W) \end{array} , \tag{2.1.7}$$

$$\begin{array}{ccc} F(\mathbb{1}) \otimes' F(V) & \xrightarrow{\phi_2(\mathbb{1}, V)} & F(\mathbb{1} \otimes V) \\ \Big\downarrow{\scriptstyle \phi_0 \otimes' \mathrm{id}_{F(V)}} & & \Big\downarrow{\scriptstyle F(\lambda_V)} \\ \mathbb{1}' \otimes' F(V) & \xrightarrow{\lambda'_{F(V)}} & F(V) \end{array} \tag{2.1.8}$$

and

$$\begin{array}{ccc} F(V)\otimes' F(\mathbb{1}) & \xrightarrow{\phi_2(V,\mathbb{1})} & F(V\otimes \mathbb{1}) \\ {\scriptstyle \mathrm{id}_{F(V)}\otimes'\phi_0}\downarrow & & \downarrow{\scriptstyle F(\rho_V)} \\ F(V)\otimes' \mathbb{1}' & \xrightarrow{\rho'_{F(V)}} & F(V) \end{array} \tag{2.1.9}$$

commute for any $U, V, W \in \mathrm{Ob}(\mathcal{C})$.

2.1.2 Duals and rigidity

Let $\mathcal{C}$ be a monoidal category, and let $X \in \mathrm{Ob}(\mathcal{C})$.

Definition 2.1.3. *An object* $X^* \in \mathrm{Ob}(\mathcal{C})$ *is called a* left dual *of* X *if there exist morphisms*

$$\mathrm{ev}^l_X : X^* \otimes X \to \mathbb{1} \tag{2.1.10}$$

and

$$\mathrm{coev}^l_X : \mathbb{1} \to X \otimes X^* \tag{2.1.11}$$

such that the composition

$$X \xrightarrow{\mathrm{coev}^l_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha^{-1}_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}^l_X} X \tag{2.1.12}$$

equals id_X*, and the composition*

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}^l_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}^l_X \otimes \mathrm{id}_{X^*}} X^* \tag{2.1.13}$$

equals id_{X^*}.

Definition 2.1.4. *An object* ${}^*X \in \mathrm{Ob}(\mathcal{C})$ *is called a* right dual *of* X *if there exist morphisms*

$$\mathrm{ev}^r_X : X \otimes {}^*X \to \mathbb{1} \tag{2.1.14}$$

and

$$\mathrm{coev}^r_X : \mathbb{1} \to {}^*X \otimes X \tag{2.1.15}$$

such that the composition

$$X \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}^r_X} X \otimes ({}^*X \otimes X) \xrightarrow{\alpha_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{\mathrm{ev}^r_X \otimes \mathrm{id}_X} X \tag{2.1.16}$$

equals id_X*, and the composition*

$${}^*X \xrightarrow{\mathrm{coev}^r_X \otimes \mathrm{id}_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{\alpha^{-1}_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathrm{id}_{{}^*X} \otimes \mathrm{ev}^r_X} {}^*X \tag{2.1.17}$$

equals $\mathrm{id}_{{}^*X}$.

Definition 2.1.5. *An object in a monoidal category is called* rigid *if it has left and right duals. A monoidal category is called* rigid *if every object in it is rigid.*

Remark 2.1.2. For a given rigid monoidal category $\mathcal{C}$, we will, generally, assume that we have fixed a choice of the morphisms $\mathrm{ev}^l_X, \mathrm{coev}^l_X, \mathrm{ev}^r_X$ and coev^r_X for any $X \in \mathrm{Ob}(\mathcal{C})$.

For any morphism $f : V \to W$, we define the *dual* of f, denoted by f^*, to be the morphism given by

$$f^* := (\mathrm{ev}^l_W \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{W^*} \otimes f \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{W^*} \otimes \mathrm{coev}^l_V) : W^* \to V^*. \qquad (2.1.18)$$

It is shown in [EGNO15] that $* : \mathcal{C} \to \mathcal{C}^{mop}$, $V \mapsto V^*$ for any object $V \in \mathrm{Ob}(\mathcal{C})$, and $f \mapsto f^*$ for any morphism $f : V \to W$, is an equivalence of monoidal categories. Here, $\mathcal{C}^{mop}$ is the opposite category of $\mathcal{C}$ (see [ML98]) equipped with the opposite tensor product. More precisely, $\mathcal{C}^{mop} = (\mathcal{C}^{op}, \otimes^{op}, \mathbb{1}, \alpha^{op}, \lambda^{op}, \rho^{op})$, where $V \otimes^{op} W = W \otimes V$, $\alpha^{op}_{U,V,W} = \alpha_{W,V,U}$, $\lambda^{op} = \rho^{-1}$, and $\rho^{op} = \lambda^{-1}$ for any $U, V, W \in \mathrm{Ob}(\mathcal{C})$.

2.1.3 Braided monoidal categories

Let $\Omega : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the component swapping functor, i.e., $\Omega(V, W) = (W, V)$ for all objects $V, W \in \mathrm{Ob}(\mathcal{C})$ and $\Omega(f, g) = (g, f)$ for all morphisms $(f, g) \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{C}}(V \times W, V' \times W')$.

Definition 2.1.6. *Let $\mathcal{C}$ be a monoidal category. A* braiding *on $\mathcal{C}$ is a natural isomorphism $\beta : \otimes \dot{\to} \otimes \circ \Omega$,*

$$\beta_{V,W} : V \otimes W \xrightarrow{\cong} W \otimes V \qquad (2.1.19)$$

natural in any pair $V, W \in \mathrm{Ob}(\mathcal{C})$, such that the following axioms are satisfied.

***Hexagon Axiom I**:*

$$
\begin{array}{ccccc}
 & (U \otimes V) \otimes W & \xrightarrow{\beta_{U\otimes V,W}} & W \otimes (U \otimes V) & \\
 {}^{\alpha_{U,V,W}}\nearrow & & & & \searrow^{\alpha_{W,U,V}} \\
U \otimes (V \otimes W) & & & & (W \otimes U) \otimes V \\
 {}_{\mathrm{id}_U \otimes \beta_{V,W}}\searrow & & & & \nearrow_{\beta_{U,W} \otimes \mathrm{id}_V} \\
 & U \otimes (W \otimes V) & \xrightarrow{\alpha_{U,W,V}} & (U \otimes W) \otimes V &
\end{array}
\tag{2.1.20}
$$

commutes for all $U, V, W \in \mathrm{Ob}(\mathcal{C})$*.*

Hexagon Axiom II*:*

$$
\begin{array}{ccccc}
 & U \otimes (V \otimes W) & \xrightarrow{\beta_{U,V\otimes W}} & (V \otimes W) \otimes U & \\
 {}^{\alpha^{-1}_{U,V,W}}\nearrow & & & & \searrow^{\alpha^{-1}_{V,W,U}} \\
(U \otimes V) \otimes W & & & & V \otimes (W \otimes U) \\
 {}_{\beta_{U,V} \otimes \mathrm{id}_W}\searrow & & & & \nearrow_{\mathrm{id}_V \otimes \beta_{U,W}} \\
 & (V \otimes U) \otimes W & \xrightarrow[\alpha^{-1}_{V,U,W}]{} & V \otimes (U \otimes W) &
\end{array}
\tag{2.1.21}
$$

commutes for any $U, V, W \in \mathrm{Ob}(\mathcal{C})$*.*

Note that the commutativity of Diagram (2.1.21) for β implies the commutativity of the Diagram (2.1.20) for β^{-1} and vice versa.

Definition 2.1.7. *A monoidal category* $\mathcal{C}$ *together a braiding* β *is called a* braided monoidal category*, or a* braided category*.*

Again, we are interested in functors between braided monoidal categories that preserve structures.

Definition 2.1.8. *Let* $\mathcal{C}, \mathcal{D}$ *be braided monoidal categories with braidings denoted by* $\beta^{\mathcal{C}}, \beta^{\mathcal{D}}$ *respectively. A monoidal functor* $(F, \phi_0, \phi_2) : \mathcal{C} \to \mathcal{D}$ *is called* braided *if the* *diagram*

$$\begin{array}{ccc} F(X)\otimes F(Y) & \xrightarrow{\beta^{\mathcal{D}}_{F(X),F(Y)}} & F(Y)\otimes F(X) \\ {\scriptstyle \phi_2(X,Y)}\downarrow & & \downarrow{\scriptstyle \phi_2(Y,X)} \\ F(X\otimes Y) & \xrightarrow{F(\beta^{\mathcal{C}}_{X,Y})} & F(Y\otimes X) \end{array} \tag{2.1.22}$$

commutes for all $X, Y \in \mathrm{Ob}(\mathcal{C})$*.*

2.1.4 Ribbon structure

Definition 2.1.9. *A* twist *on a braided, rigid, monoidal category* $\mathcal{C}$ *is a natural isomorphism* $\theta : \mathrm{id}_{\mathcal{C}} \dot{\to} \mathrm{id}_{\mathcal{C}}$ *such that*

$$\theta_{V\otimes W} = (\theta_V \otimes \theta_W) \circ \beta_{W,V} \circ \beta_{V,W} \tag{2.1.23}$$

for all $V, W \in \mathrm{Ob}(\mathcal{C})$*. A twist is called a* ribbon structure *if*

$$(\theta_V)^* = \theta_{V^*} \tag{2.1.24}$$

for any $V \in \mathrm{Ob}(\mathcal{C})$*.*

Let $\mathcal{C}$ be a braided, rigid, monoidal category with a ribbon structure θ. For any $V \in \mathrm{Ob}(\mathcal{C})$, define the *Drinfeld morphism* $\nu_V : V \to V^{**}$ by

$$V \xrightarrow{\mathrm{id}_V \otimes \mathrm{coev}_{V^*}} V \otimes V^* \otimes V^{**} \xrightarrow{\beta_{V,V^*}\otimes \mathrm{id}_{V^{**}}} V^* \otimes V \otimes V^{**} \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{V^{**}}} V^{**}. \tag{2.1.25}$$

Note that we ignore the parentheses and suppressed the associativity constraints here (see Remark 2.1.1). The Drinfeld morphism ν is natural in V and it is actually an isomorphism, for example, see [EGNO15, Section 8.10] for a proof. Let Ψ be the natural isomorphism defined by the composition of the Drinfeld morphism and the ribbon structure

$$\Psi = \nu \circ \theta. \tag{2.1.26}$$

By the discussion above, for any $V \in \mathrm{Ob}(\mathcal{C})$, V is isomorphic to V^{**} via the isomorphism $\Psi_V : V \xrightarrow{\cong} V^{**}$. Note that $\Psi_{V\otimes W} = \Psi_V \otimes \Psi_W$ (using Equation (2.1.23)), but $\nu_{V\otimes W}$ is not equal to $\nu_V \otimes \nu_W$.

Let $\mathcal{C}$ be a braided rigid monoidal category with the ribbon structure θ. Let $\Psi = \nu \circ \theta$ be the above natural isomorphism.

Definition 2.1.10. *For any $V \in \mathrm{Ob}(\mathcal{C})$ and any $f \in \mathrm{End}_{\mathcal{C}}(V) = \mathrm{Hom}_{\mathcal{C}}(V, V)$, the* quantum trace *of f, denoted by $\mathrm{tr}\,(f)$, is an element in $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ defined by the following composition of morphisms:*

$$\mathrm{tr}\,(f) : \mathbb{1} \xrightarrow{\mathrm{coev}_V} V \otimes V^* \xrightarrow{f\otimes \mathrm{id}_{V^*}} V \otimes V^* \xrightarrow{\Psi_V\otimes \mathrm{id}_{V^*}} V^{**} \otimes V^* \xrightarrow{\mathrm{ev}_{V^*}} \mathbb{1}. \tag{2.1.27}$$

Definition 2.1.11. *The* quantum dimension *of $V \in \mathrm{Ob}(\mathcal{C})$, denoted by d_V, is defined to be the quantum trace of id_V. In other words,*

$$d_V = \mathrm{tr}\,(\mathrm{id}_V) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}). \tag{2.1.28}$$

2.2 Modular categories

2.2.1 Fusion categories

Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category. For detailed definitions of such a category as well as the definition of the zero object and subobjects, the reader is referred to

[EGNO15] Chapter 1 or [BK01] Chapter 1.

Definition 2.2.1. *An object* $a \in \mathrm{Ob}(\mathcal{C})$ *is called* simple *if 0 and* a *itself are its only subobjects. We call* $\mathcal{C}$ semisimple *if every object* $b \in \mathrm{Ob}(\mathcal{C})$ *is isomorphic to a direct sum of simple objects.*

From each isomorphism class of simple objects of $\mathcal{C}$, we choose a representative. The set of the chosen representatives of isomorphism classes of simple objects in $\mathcal{C}$ is denoted by $\mathrm{Irr}(\mathcal{C})$.

Semisimplicity can be expressed by the following formula for any $b \in \mathrm{Ob}(\mathcal{C})$:

$$b \cong \bigoplus_{a \in \mathrm{Irr}(\mathcal{C})} M(b,a)a. \tag{2.2.1}$$

Here, $M(b,a) \in \mathbb{Z}_{\geq 0}$, and there are only finitely many $a \in \mathrm{Irr}(\mathcal{C})$ such that $M(b,a) \neq 0$.

Besides Equation (2.2.1), in this thesis, we mainly use the following facts about a finite semisimple $\mathbb{C}$-linear abelian category $\mathcal{C}$:

1. Spaces of morphisms (or Hom-spaces) of $\mathcal{C}$ are finite dimensional vector spaces over $\mathbb{C}$;

2. In Equation (2.2.1), $M(b,a) = \dim(\mathrm{Hom}_{\mathcal{C}}(b,a))$;

3. There are finitely many isomorphism classes of simple objects. Let $|\mathrm{Irr}(\mathcal{C})|$ be the cardinality of $\mathrm{Irr}(\mathcal{C})$, we have $|\mathrm{Irr}(\mathcal{C})| < \infty$;

4. (Schur's Lemma.) For any $a, b \in \mathrm{Irr}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}}(a,b) = \delta_{a,b}\mathbb{C}. \tag{2.2.2}$$

Definition 2.2.2. *Let $\mathcal{C}$ be a finite semisimple $\mathbb{C}$-linear abelian category. We call $\mathcal{C}$ a* fusion category *if $\mathcal{C}$ is also a rigid monoidal category such that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on morphisms and $\mathbb{1} \in \mathrm{Irr}(\mathcal{C})$.*

Natural examples of fusion categories include ***Vec***, the category of finite dimensional vector spaces over $\mathbb{C}$ with the tensor product given by the vector space tensor product. The tensor unit $\mathbb{1}$ is $\mathbb{C}$, the one-dimensional vector space. The category ***Rep***(G) of finite dimensional representations over $\mathbb{C}$ of a finite group G is also a fusion category with the tensor unit being the trivial representation.

Let $\mathcal{C}$ be a fusion category. Rigidity guarantees that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is biexact. More precisely, it is shown in [BK01, Proposition 2.1.8] that for any short exact sequence

$$0 \to a \to b \to c \to 0 \tag{2.2.3}$$

in $\mathcal{C}$, and any $j \in \mathrm{Ob}(\mathcal{C})$, the sequences

$$\begin{gathered} 0 \to a \otimes j \to b \otimes j \to c \otimes j \to 0 \\ 0 \to j \otimes a \to j \otimes b \to j \otimes c \to 0 \end{gathered} \tag{2.2.4}$$

are exact.

Definition 2.2.3. *The* Grothendieck group *$K_0(\mathcal{C})$ of an abelian category $\mathcal{C}$ is the quotient of the free abelian group on the set of all isomorphism classes of objects in $\mathcal{C}$ modulo the relations $\langle b \rangle = \langle a \rangle + \langle c \rangle$ for any short exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{C}$, where $\langle a \rangle$ stands for the isomorphism class of a. If $\mathcal{C}$ is a fusion category, the tensor product endows $K_0(\mathcal{C})$ with a ring structure whose multiplication is given by $\langle a \rangle \langle b \rangle = \langle a \otimes b \rangle$. We will call the ring $K_0(\mathcal{C})$ the* Grothendieck ring *of $\mathcal{C}$.*

Note that the multiplication is well-defined when the tensor product is biexact.

Note also that as abelian groups,

$$K_0(\mathcal{C}) \cong \mathbb{Z}^{\mathrm{Irr}(\mathcal{C})}. \tag{2.2.5}$$

Let $\mathcal{C}$ be a fusion category. Then a ring structure of $K_0(\mathcal{C})$ is determined by the following data. For any $a, b \in \mathrm{Irr}(\mathcal{C})$, Equation (2.2.1) implies that

$$\langle a\rangle\langle b\rangle = \langle a \otimes b\rangle = \sum_{c\in\mathrm{Irr}(\mathcal{C})} N_{ab}^{c}\langle c\rangle, \tag{2.2.6}$$

where $N_{ab}^{c} = \dim(\mathrm{Hom}_{\mathcal{C}}(a \otimes b, c))$. In practice, we will also write

$$a \otimes b = \bigoplus_{c\in\mathrm{Irr}(\mathcal{C})} N_{ab}^{c} c \tag{2.2.7}$$

by abuse of notation, and we will call both Equations (2.2.6) and (2.2.7) the *fusion rules* of $\mathcal{C}$. N_{ab}^{c} are called *fusion coefficients*. A triple (a, b, c) of simple objects in $\mathcal{C}$ is called *admissible* if $N_{ab}^{c} \neq 0$.

Definition 2.2.4. *A fusion category $\mathcal{C}$ is called* multiplicity-free *if for any $a, b, c \in \mathrm{Irr}(\mathcal{C})$, N_{ab}^{c} equals either 0 or 1.*

It can be shown that if $\mathcal{C}$ is a fusion category, then for any object $a \in \mathrm{Ob}(\mathcal{C})$, there are isomorphisms ${}^*a \cong a^*$, and $a \cong a^{**}$. (For a proof, see [EGNO15] Proposition 4.8.1). Hence when we are working with fusion categories, we don't distinguish the isomorphism classes of the left and right duals of an object.

Definition 2.2.5. *A fusion category $\mathcal{C}$ is called* self-dual *if for all $a \in \mathrm{Ob}(\mathcal{C})$, $a \cong a^*$.*

2.2.2 Modular categories

Definition 2.2.6. *A fusion category $\mathcal{C}$ equipped with a braiding β is called a* braided fusion category. *A braided fusion category equipped with a ribbon structure* θ *is called a* ribbon fusion category*.*

The ribbon categories in the rest of this thesis will all be understood as ribbon fusion categories.

Let $\mathcal{C}$ be a ribbon category with the ribbon structure θ. Note that by Schur's Lemma, for any simple object $a \in \mathrm{Irr}(\mathcal{C})$,

$$\theta_a \in \mathrm{Hom}_{\mathcal{C}}(a, a) \cong \mathbb{C} \tag{2.2.8}$$

is a non-zero scalar multiple of id_a. By abuse of notation, we will denote this scalar again by θ_a.

Note also that in a ribbon category $\mathcal{C}$, Schur's lemma implies that the quantum trace of any endomorphism, in particular, the quantum dimension of any object in $\mathcal{C}$, is in $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \mathbb{C}$.

Recall that for a fusion category $\mathcal{C}$, $|\mathrm{Irr}(\mathcal{C})| < \infty$.

Definition 2.2.7. *Let* $\mathcal{C}$ *be a ribbon fusion category. For any* $a, b \in \mathrm{Irr}(\mathcal{C})$*, define*

$$\tilde{\mathfrak{s}}_{ab} := \mathrm{tr}\,(\beta_{b,a} \circ \beta_{a,b}) \in \mathbb{C} = \mathrm{End}_{\mathcal{C}}(\mathbb{1}). \tag{2.2.9}$$

$\mathcal{C}$ *is called a* modular category *if the* $(|\mathrm{Irr}(\mathcal{C})| \times |\mathrm{Irr}(\mathcal{C})|)$*-matrix*

$$\tilde{\mathfrak{s}} = (\tilde{\mathfrak{s}}_{ab})_{a,b \in \mathrm{Irr}(\mathcal{C})} \tag{2.2.10}$$

is invertible. We call $\tilde{\mathfrak{s}}$ *the* unnormalized S-matrix*.*

The reason we call the ribbon categories satisfying the above condition modular is because their relationship to the modular group $\mathrm{SL}(2, \mathbb{Z})$, as is explained in the following.

It is shown in [ENO05, Section 2.1] that in a ribbon category $\mathcal{C}$, $d_a \in \mathbb{R}$, and $d_a^2 > 0$ for any $a \in \mathrm{Irr}(\mathcal{C})$. Let

$$\dim(\mathcal{C}) := \sum_{a \in \mathrm{Irr}(\mathcal{C})} d_a^2 \tag{2.2.11}$$

be the the *global dimension* of $\mathcal{C}$. In particular, $\dim(\mathcal{C}) \neq 0$. Denote the positive square root of $\dim(\mathcal{C})$ by $\mathcal{D}$

$$\mathcal{D} := \sqrt{\dim(\mathcal{C})}. \tag{2.2.12}$$

We call the matrix

$$\mathfrak{s} := \frac{1}{\mathcal{D}} \cdot \tilde{\mathfrak{s}} \tag{2.2.13}$$

the (normalized) *S-matrix*.

Consider another $(|\mathrm{Irr}(\mathcal{C})| \times |\mathrm{Irr}(\mathcal{C})|)$-matrix $\mathfrak{t}$ with entries

$$\mathfrak{t}_{jk} := \delta_{j,k}\theta_j \tag{2.2.14}$$

for all $j, k \in \mathrm{Irr}(\mathcal{C})$. We call $\mathfrak{t}$ the *T-matrix* of $\mathcal{C}$.

The reason we consider $\mathfrak{s}$ and $\mathfrak{t}$ is that they are important invariants of the category $\mathcal{C}$. Let

$$\hat{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.2.15}$$

and

$$\hat{\tau} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.2.16}$$

be the standard generators of $\mathrm{SL}(2,\mathbb{Z})$.

For any finite dimensional complex vector space V, let $M \in \mathrm{GL}(V)$ be an automorphism of V. Let $\mathrm{PGL}(V)$ be the quotient of $\mathrm{GL}(V)$ by its center. We denote the equivalence class of M in the group $\mathrm{PGL}(V)$ by $\{M\}$.

It is well-known (see, for example [BK01]) that when $\mathcal{C}$ is a modular category, the map

$$\mu : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PGL}(K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}) \tag{2.2.17}$$

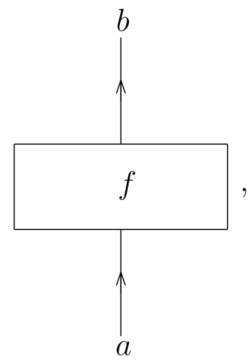
sending $\hat{\sigma}$ to $\{\mathfrak{s}\}$ and $\hat{\tau}$ to $\{\mathfrak{t}\}$ is a group homomorphism. In other words, μ is a *projective representation* of the modular group $\mathrm{SL}(2,\mathbb{Z})$. The S-matrix and the T-matrix are usually referred to as the *modular data* of the category $\mathcal{C}$. It is worth noting that $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ has the structure of a finite dimensional semisimple $\mathbb{C}$-algebra. The interested reader is referred to [Lus87]. Algebraic properties of the modular representation μ will be discussed in the following chapters.

2.3 Graphical calculus for modular categories

2.3.1 Introduction to graphical calculus

In this section we introduce a graphical calculus as a useful tool to represent morphisms in a modular category $\mathcal{C}$ [RT91]. From now on we will assume that $\mathcal{C}$ is **multiplicity-free** with structural morphisms $\alpha, \beta, \lambda, \rho, \theta$ as introduced in Chapter 1.

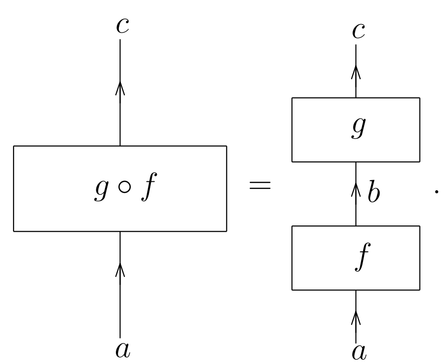
We represent a morphism $f : a \to b$ by the picture



(2.3.1)

reading from bottom to top. We can think of the directed strings colored by a and b as the source and target of f. The meaning of the directions of the strings will be explained later. The box colored by f (or "f-colored coupon") keeps track of the name of the morphism.

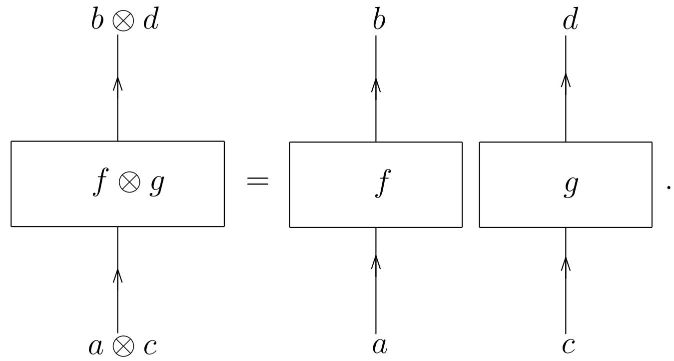
Composition of two morphisms is represented by stacking one diagram on top of another. More precisely, if $f : a \to b$, $g : b \to c$ we have the following pictorial equivalence:



(2.3.2)

Note that two morphisms are composable if and only if the colors of their source and target strands match.

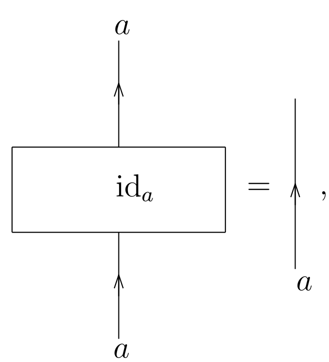
Tensor products of morphisms are given by horizontal juxtaposition. If $f : a \to b$, $g : c \to d$, then



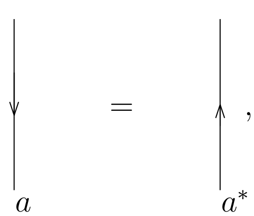
(2.3.3)

By Mac Lane's Coherence Theorem, we can suppress the associativity constraint α (Remark 2.1.1).

To special morphisms we assign special pictures. We view a single strand colored by an object pointing upwards (as indicated by the arrow on the strand) both as the identity morphism of the object and as the object itself, and we view a single strand colored by a pointing downwards as the identity morphism of a^*. For simplicity, we usually omit the tensor unit object or denote it by a dashed line. The pictures for the above conventions are

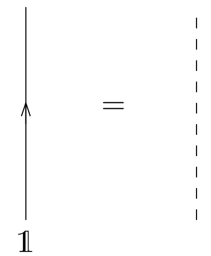


(2.3.4)



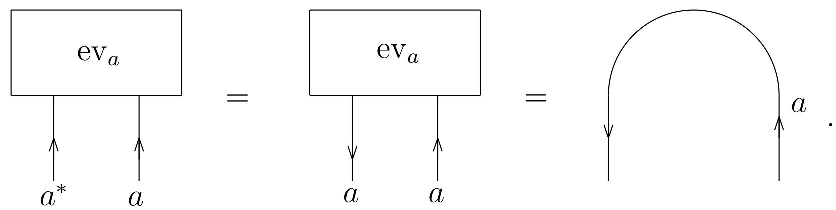
(2.3.5)

and



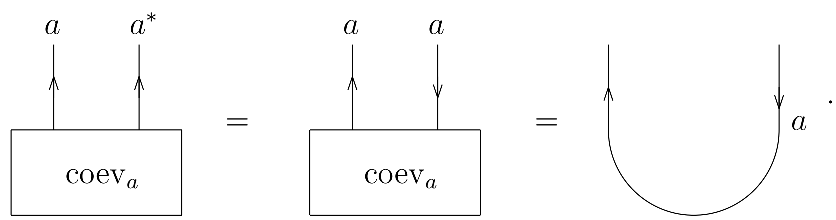
(2.3.6)

Evaluation and coevaluation are given by a cup and a cap "bending left" (oriented left at the extrema):



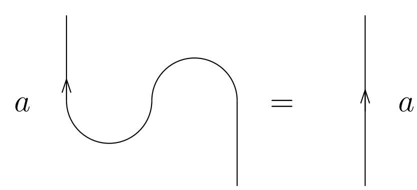
(2.3.7)

Similarly,



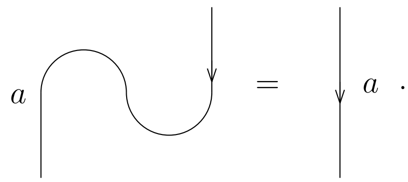
(2.3.8)

In this way, rigidity conditions become topological moves

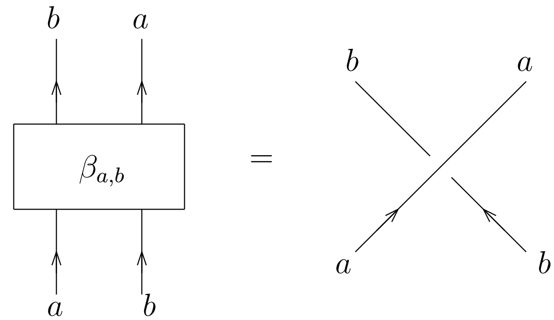


(2.3.9)

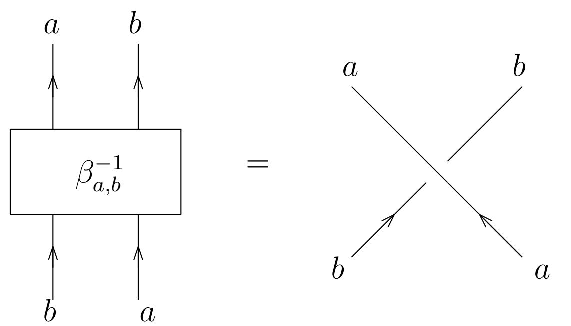
and

 (2.3.10)

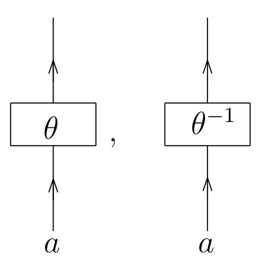
We denote braidings by

 (2.3.11)

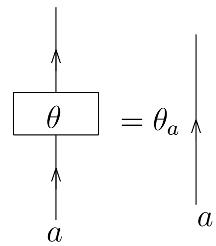
and

 (2.3.12)

We denote the twist and its inverse by

 (2.3.13)

As discussed in the last section, for a simple object $a \in \mathrm{Irr}(\mathcal{C})$, θ_a acts a scalar, which, by abuse of notation, is denoted again by θ_a. In graphical calculus, we write

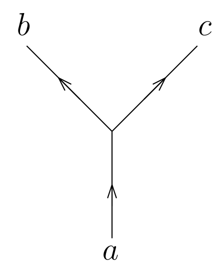


(2.3.14)

2.3.2 A gift from linearity: basis vectors in Hom-spaces

Hom-spaces in $\mathcal{C}$ are finite dimensional $\mathbb{C}$-vector spaces. In this section, we give graphical presentations of the basis vectors. By Schur's lemma, for all simple object a, we pick id_a as the basis vector of the 1-dimensional vector space $\mathrm{Hom}_{\mathcal{C}}(a, a)$.

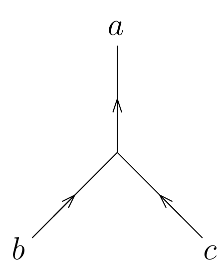
By assumption, $\mathcal{C}$ is multiplicity-free, which means that for all admissible simple objects a, b, c, $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c)$ and $\mathrm{Hom}_{\mathcal{C}}(b \otimes c, a)$ are 1-dimensional. We will choose a basis vector for each $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c)$, and denote the basis vector by a trivalent vertex



. (2.3.15)

For completeness, we adopt the convention that if a, b, c are not admissible, then the trivalent vertex colored by any order of a, b, c is understood as 0.

We denote a chosen basis $\mathrm{Hom}_{\mathcal{C}}(b \otimes c, a)$ by the trivalent vertex

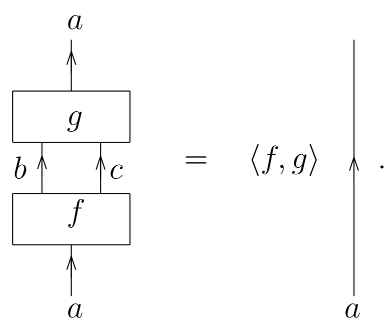


. (2.3.16)

By ([Tur10, Lemma II.4.2.3]), there is a non-degenerate bilinear pairing

$$\begin{array}{rlcc} \langle \cdot, \cdot \rangle : & \mathrm{Hom}_{\mathcal{C}}(a, b \otimes c) \otimes \mathrm{Hom}_{\mathcal{C}}(b \otimes c, a) & \longrightarrow & \mathbb{C} \\ & f \otimes g & \mapsto & \dfrac{\mathrm{tr}\,(g \circ f)}{d_a} \end{array} \tag{2.3.17}$$

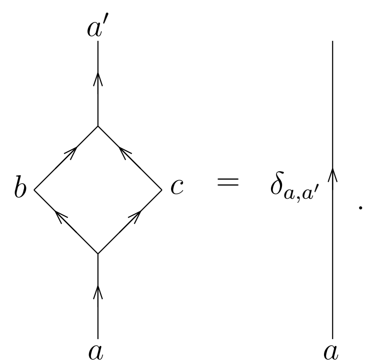
between the Hom-spaces. In terms of graphical calculus, we have



(2.3.18)

In fact, we can define the form $\langle \cdot, \cdot \rangle$ between any non-zero Hom-spaces $\mathrm{Hom}_{\mathcal{C}}(V, W)$ and $\mathrm{Hom}_{\mathcal{C}}(W, V)$ for objects $V, W \in \mathrm{Ob}(\mathcal{C})$. Therefore, we can identify $\mathrm{Hom}_{\mathcal{C}}(W, V)$ as the dual space of $\mathrm{Hom}_{\mathcal{C}}(V, W)$ (note that the dual here is taken in the category of vector spaces).

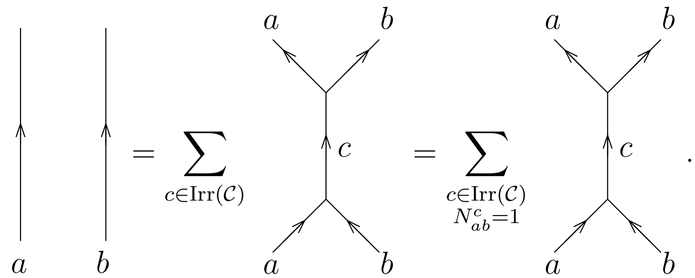
We normalize the trivalent vertex bases of $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c)$ and $\mathrm{Hom}_{\mathcal{C}}(b \otimes c, a)$ such that



(2.3.19)

We often refer to this formula as the *box-elimination formula*, or the bigon relation.

Lemma 2.3.1 (Fusion-splitting formula)**.** *For any pair of simple objects $a, b \in \mathrm{Irr}(\mathcal{C})$,*



(2.3.20)

Note that if $\mathrm{Hom}_{\mathcal{C}}(a \otimes b, c) = 0$, the morphism represented by the picture summand in the middle is the 0 morphism, so we have the second equality for free.

Proof. It is enough to show that the first term is equal to the third term.

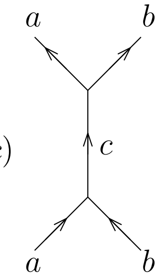
Since $\mathcal{C}$ is multiplicity-free, we have

$$a \otimes b = \bigoplus_{\substack{c \in \mathrm{Irr}(\mathcal{C}) \\ N_{ab}^c = 1}} c. \tag{2.3.21}$$

By the definition of direct sums in linear (or more generally, abelian) categories [ML98], there exist $f_c \in \mathrm{Hom}_{\mathcal{C}}(a \otimes b, c)$ and $g_c \in \mathrm{Hom}_{\mathcal{C}}(c, a \otimes b)$ for every c such that

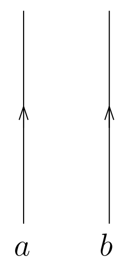
$$\mathrm{id}_{a \otimes b} = \sum_{\substack{c \in \mathrm{Irr}(\mathcal{C}) \\ N_{ab}^c = 1}} g_c \circ f_c. \tag{2.3.22}$$

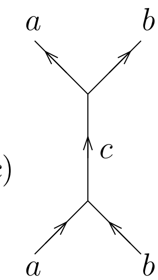
Since $\mathcal{C}$ is multiplicity-free, we also have $\mathrm{Hom}_{\mathcal{C}}(a \otimes b, c) \otimes \mathrm{Hom}_{\mathcal{C}}(c, a \otimes b)$ is at most one-dimensional. Hence, for any $c \in \mathrm{Irr}(\mathcal{C})$ such that $N_{ab}^c = 1$, there exists a complex number $\mu(a, b, c) \neq 0$ such that

$$g_c \circ f_c = \mu(a,b,c)$$ 

$$. \qquad (2.3.23)$$

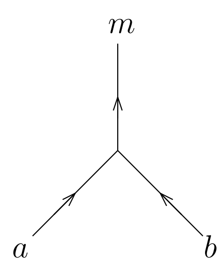
Equation (2.3.22) now becomes

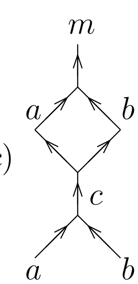


$$= \sum_{\substack{c \in \mathrm{Irr}(\mathcal{C}) \\ N_{ab}^{c}=1}} \mu(a,b,c)$$ 

$$. \qquad (2.3.24)$$

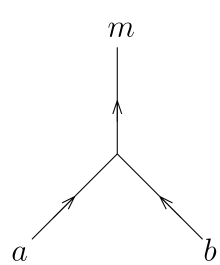
For any $m \in \mathrm{Irr}(\mathcal{C})$ such that $N_{ab}^{m} = 1$, composing both sides of the above equation with the trivalent vertex basis for $\mathrm{Hom}_{\mathcal{C}}(a \otimes b, m)$, we have

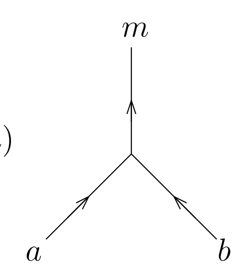


$$= \sum_{\substack{c \in \mathrm{Irr}(\mathcal{C}) \\ N_{ab}^{c}=1}} \mu(a,b,c)$$ 

$$. \qquad (2.3.25)$$

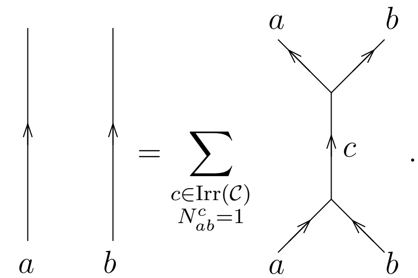
By Equation (2.3.19), we have



$$= \mu(a,b,m)$$ 

$$. \qquad (2.3.26)$$

Therefore, for any simple object m such that $N_{ab}^{m} = 1$, we have $\mu(a,b,m) = 1$. Now Equation (2.3.24) becomes



(2.3.27)

□

2.3.3 Structural constants: F-matrices

For any simple objects $a, b, c, l \in \mathrm{Irr}(\mathcal{C})$, consider an isomorphism $\mathcal{F}^a_{bcl}$ defined by requiring the following diagram of isomorphisms to be commutative:

$$\begin{array}{ccc} \bigoplus_{m\in\mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(a, b\otimes m)\otimes \mathrm{Hom}_{\mathcal{C}}(m, c\otimes l) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}}(a, b\otimes(c\otimes l)) \\ \downarrow \mathcal{F}^a_{bcl} & & \downarrow \mathrm{Hom}_{\mathcal{C}}(a,\alpha_{b,c,l}) \\ \bigoplus_{n\in\mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(a, n\otimes l)\otimes \mathrm{Hom}_{\mathcal{C}}(n, b\otimes c) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}}(a, (b\otimes c)\otimes l) \end{array}, \quad (2.3.28)$$

where the horizontal isomorphisms are given by (see [Tur10] Lemma VI.1.1.1 and VI.1.1.2)

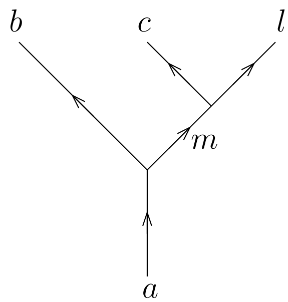
$$(f, g) \mapsto (\mathrm{id}_b \otimes g)\circ f \qquad (2.3.29)$$

and

$$(f', g') \mapsto (g' \otimes \mathrm{id}_l)\circ f' \qquad (2.3.30)$$

respectively. Note that $\mathrm{Hom}_{\mathcal{C}}(a, \cdot) : \mathcal{C} \to \boldsymbol{Vec}$ is a covariant functor. Since $\alpha_{b,c,l}$ is an isomorphism, $\mathrm{Hom}_{\mathcal{C}}(a, \alpha_{b,c,l})$ is an isomorphism.

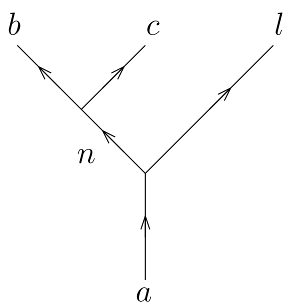
Therefore, if $\operatorname{Hom}_{\mathcal{C}}(a, b \otimes c \otimes l) \neq 0$, we have bases for the $\mathbb{C}$-vector spaces $\bigoplus_{m \in \operatorname{Irr}(\mathcal{C})} \operatorname{Hom}_{\mathcal{C}}(a, b \otimes m) \otimes \operatorname{Hom}_{\mathcal{C}}(m, c \otimes l)$ and $\bigoplus_{n \in \operatorname{Irr}(\mathcal{C})} \operatorname{Hom}_{\mathcal{C}}(a, n \otimes l) \otimes \operatorname{Hom}_{\mathcal{C}}(n, b \otimes c)$ respectively built from the trivalent vertex basis in the previous section. More precisely, we have the *right branched basis*

$\Bigg\{$ 

$: \ m \in \operatorname{Irr}(\mathcal{C}),\ \operatorname{Hom}_{\mathcal{C}}(a, b \otimes m) \otimes \operatorname{Hom}_{\mathcal{C}}(m, c \otimes l) \neq 0 \Bigg\}$

(2.3.31)

for $\bigoplus_{m \in \operatorname{Irr}(\mathcal{C})} \operatorname{Hom}_{\mathcal{C}}(a, b \otimes m) \otimes \operatorname{Hom}_{\mathcal{C}}(m, c \otimes l)$, and *left branched basis*

$\Bigg\{$ 

$: \ n \in \operatorname{Irr}(\mathcal{C}),\ \operatorname{Hom}_{\mathcal{C}}(a, b \otimes n) \otimes \operatorname{Hom}_{\mathcal{C}}(n, c \otimes l) \neq 0 \Bigg\}$ (2.3.32)

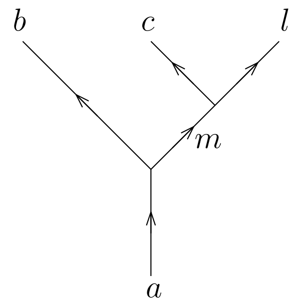
for $\bigoplus_{n \in \operatorname{Irr}(\mathcal{C})} \operatorname{Hom}_{\mathcal{C}}(a, n \otimes l) \otimes \operatorname{Hom}_{\mathcal{C}}(n, b \otimes c)$.

Recall that in the graphical calculus, we usually omit the associativity constraints α .

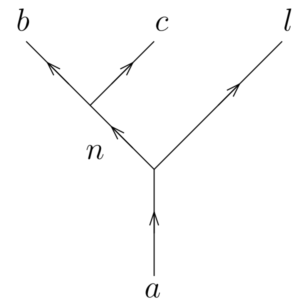
Definition 2.3.1. *For simple objects $a, b, c, l \in \operatorname{Irr}(\mathcal{C})$, if $\operatorname{Hom}_{\mathcal{C}}(a, b \otimes c \otimes l) \neq 0$, we define the* F-matrix F^a_{bcl} *to be the matrix presentation of the isomorphism $\mathcal{F}^a_{bcl}$ defined in Diagram (2.3.28), with respect to the left and right branched basis defined above.*

We usually view the left and right branched bases as bases for $\operatorname{Hom}_{\mathcal{C}}(a, b \otimes c \otimes l)$

via the horizontal isomorphisms in Diagram (2.3.28), and, therefore, we view the F-matrix as a change of basis matrix in $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c \otimes l)$. Graphically, we have

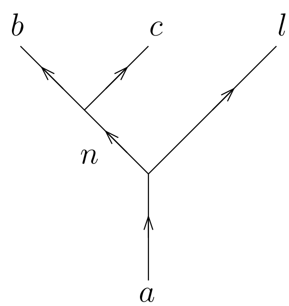


$$= \sum_{n \in \mathrm{Irr}(\mathcal{C})} (F^a_{bcl})_{nm}$$

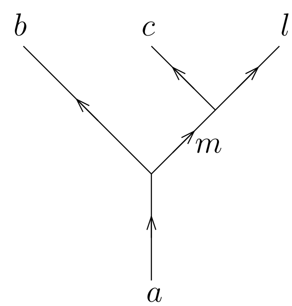


$$. \qquad (2.3.33)$$

We denote the inverse of the F-matrix F^a_{bcl} by G^a_{bcl}. In other words,



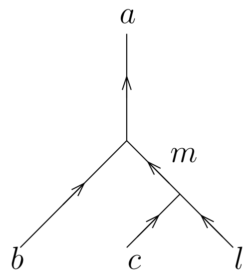
$$= \sum_{n \in \mathrm{Irr}(\mathcal{C})} (G^a_{bcl})_{mn}$$



$$. \qquad (2.3.34)$$

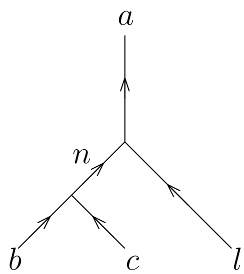
Once we pick a basis for each $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c \otimes l)$ (either left or right branched), we can simply choose the basis for $\mathrm{Hom}_{\mathcal{C}}(b \otimes c \otimes l, a)$ to be the respective dual basis via the non-degenerate bilinear form extending the one in Equation (2.3.17). We will call the dual basis to a left (right) branched basis a *left (right) fused basis*.

Graphically, the right fused basis for $\mathrm{Hom}_{\mathcal{C}}(b \otimes c \otimes l, a)$ is given by



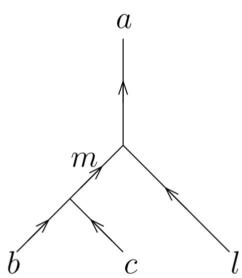
$$\left\{ \quad : \; m \in \mathrm{Irr}(\mathcal{C}), \; \mathrm{Hom}_{\mathcal{C}}(c \otimes l, m) \otimes \mathrm{Hom}_{\mathcal{C}}(b \otimes m, a) \neq 0 \right\}, \quad (2.3.35)$$

and the left fused basis for $\mathrm{Hom}_{\mathcal{C}}(b \otimes c \otimes l, a)$ is given by

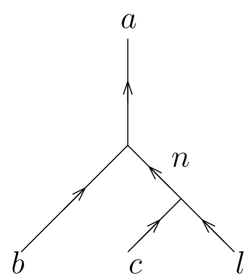


$$\left\{ \quad : \; n \in \mathrm{Irr}(\mathcal{C}), \; \mathrm{Hom}_{\mathcal{C}}(b \otimes c, n) \otimes \mathrm{Hom}_{\mathcal{C}}(n \otimes l, a) \neq 0 \right\}. \quad (2.3.36)$$

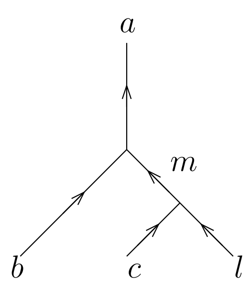
Using the bilinear pairing Equation (2.3.17) between the spaces $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c \otimes l)$ and $\mathrm{Hom}_{\mathcal{C}}(b \otimes c \otimes l, a)$, we have



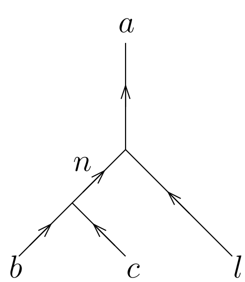
$$= \sum_{n \in \mathrm{Irr}(\mathcal{C})} (F^{a}_{bcl})_{mn} \quad (2.3.37)$$



and



$$= \sum_{n \in \mathrm{Irr}(\mathcal{C})} (G^{a}_{bcl})_{mn} \quad (2.3.38)$$



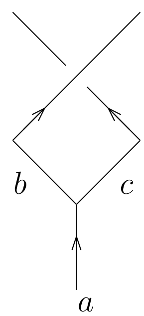
.

In practice, we will use both left and right branched basis depending on the problem, and we change the bases using F-matrices.

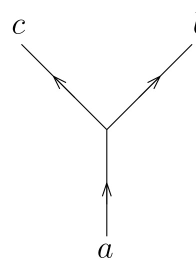
2.3.4 Structural constants: R-matrices

We can also compose braidings with the basis vector in $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c)$ for simple objects a, b, c. Recall that by the multiplicity-free assumption, if $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c) \neq 0$, it is 1-dimensional.

Definition 2.3.2. *Let* $a, b, c \in \mathrm{Irr}(\mathcal{C})$ *such that* $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c) \neq 0$. *We define the* R-matrix R^a_{bc} *to be the scalar such that*



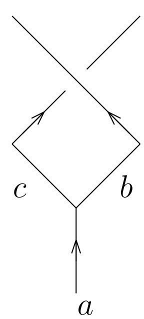
$= R^a_{bc}$



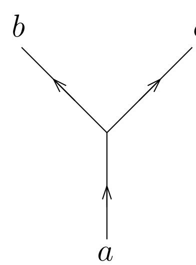
. (2.3.39)

Remark 2.3.1. R^a_{bc} should be viewed as a (1×1)-matrix. When $\mathcal{C}$ is not multiplicity-free, the R-matrix will be an $(N^a_{bc} \times N^a_{bc})$-matrix.

By the above definition and the fact that $\beta^{-1}_{b,c} \circ \beta_{b,c} = \mathrm{id}_{b \otimes c}$, we immediately have

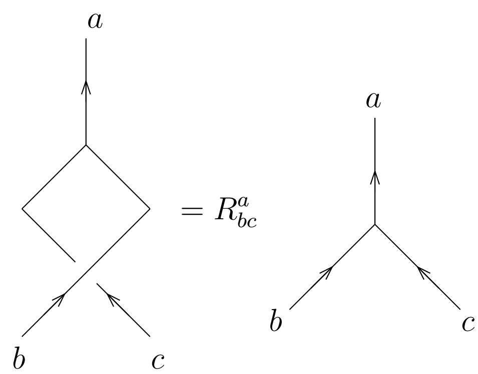


$= (R^a_{bc})^{-1}$



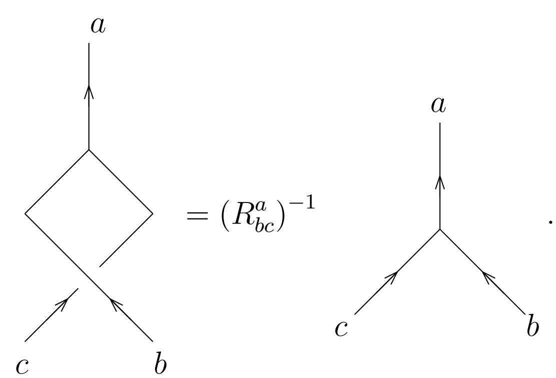
. (2.3.40)

Moreover, by identifying $\mathrm{Hom}_{\mathcal{C}}(b \otimes c, a)$ with the dual space of $\mathrm{Hom}_{\mathcal{C}}(a, b \otimes c)$ via the pairing (2.3.17), we have



(2.3.41)

and

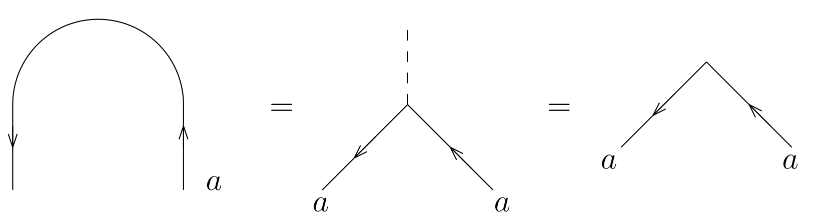


(2.3.42)

2.3.5 Evaluation and coevaluation

For any simple object $a \in \mathrm{Ob}(\mathcal{C})$, there are fixed special morphisms in the 1-dimensional $\mathbb{C}$-vector spaces $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, a \otimes a^*)$ and $\mathrm{Hom}_{\mathcal{C}}(a^* \otimes a, \mathbb{1})$, namely, the evaluation and coevaluation morphisms. In this section, we relate evaluation and coevaluation to the basis vectors in the corresponding Hom-spaces. Let $a \in \mathrm{Irr}(\mathcal{C})$ be a simple object.

For the evaluation, we set

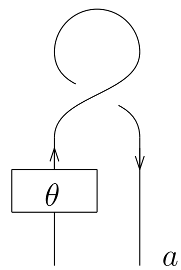


(2.3.43)

Consider the morphism

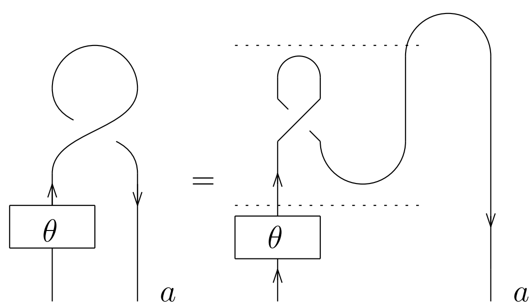
$$a \otimes a^* \xrightarrow{\theta_a \otimes \mathrm{id}_{a^*}} a \otimes a^* \xrightarrow{\beta_{a,a^*}} a^* \otimes a \xrightarrow{\mathrm{ev}_a} \mathbb{1}. \tag{2.3.44}$$

Pictorially, it is given by



. (2.3.45)

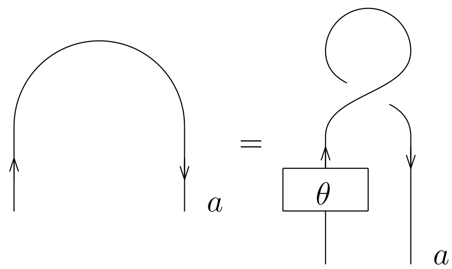
Note that by rigidity, we have



, (2.3.46)

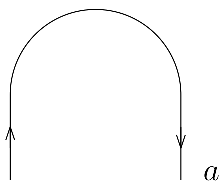
where the part of the picture between the dotted lines represents the Drinfeld morphism ν_a (cf. Equation (2.1.25)). Therefore, the part of the picture starting from the left end up to the top dotted line stands for the isomorphism $\Psi_a = \nu_a \circ \theta_a : a \to a^{**}$ identifying a with a^{**} (cf. Equation (2.1.26)).

It is shown (see, for example, [Tur10, Kas95]) that there is a choice of rigidity morphisms such that the above morphism is exactly ev_{a^*} when we identify a with a^{**} via the isomorphism Ψ_a, and we will fix this choice. In other words,

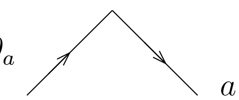


$$.\qquad (2.3.47)$$

Combining Equations (2.3.13, 2.3.42) and (2.3.43), we have



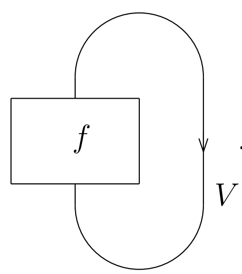
$$= R^{\mathbb{1}}_{aa^*}\theta_a$$



$$.\qquad (2.3.48)$$

By Definition (2.1.10) and the above discussion about the graphical calculus presentation of Ψ, for any $V \in \mathrm{Ob}(\mathcal{C})$ and any $f \in \mathrm{End}_{\mathcal{C}}(V)$, the quantum trace of f is given by

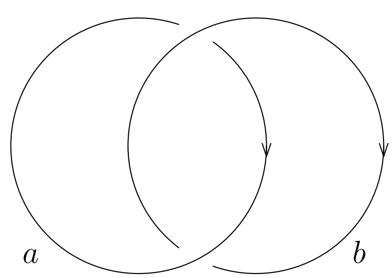
$$\mathrm{tr}\,(f) =$$



$$.\qquad (2.3.49)$$

In particular, the unnormalized S-matrix entries (see Equation (2.2.9)) can be given graphically as the value of the Hopf link colored by the corresponding simple objects:

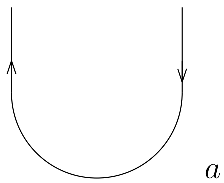
$$\tilde{\mathfrak{s}}_{ab} =$$



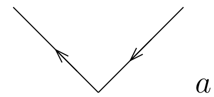
$$.\qquad (2.3.50)$$

Once we have the above normalization for evaluations, the normalization for coevaluations cannot be arbitrarily chosen. More precisely, we have the following lemma.

Lemma 2.3.2. *For any* $a \in \mathrm{Irr}(\mathcal{C})$, *we have*



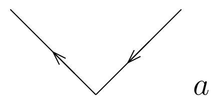
$$= \frac{d_a}{R^{\mathbb{1}}_{aa^*}\theta_a}$$



. (2.3.51)

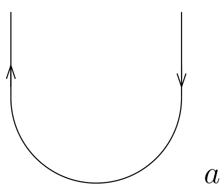
Note that since the braiding and the twists are isomorphisms, $R^{\mathbb{1}}_{aa^*}\theta_a \neq 0$.

Proof. Since $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, a \otimes a^*)$ is one-dimensional with the fixed basis

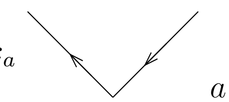


, (2.3.52)

we have



$$= x_a$$

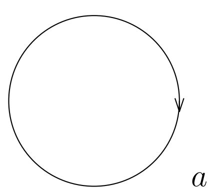


(2.3.53)

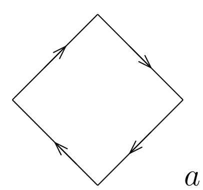
for some $x_a \in \mathbb{C}$.

By the definition of the quantum dimension, we have

$$d_a = \mathrm{tr}\,(\mathrm{id}_a) =$$



$$= x_a R^{\mathbb{1}}_{aa^*}\theta_a$$



(2.3.54)

$$= x_a R^{\mathbb{1}}_{aa^*}\theta_a.$$

□

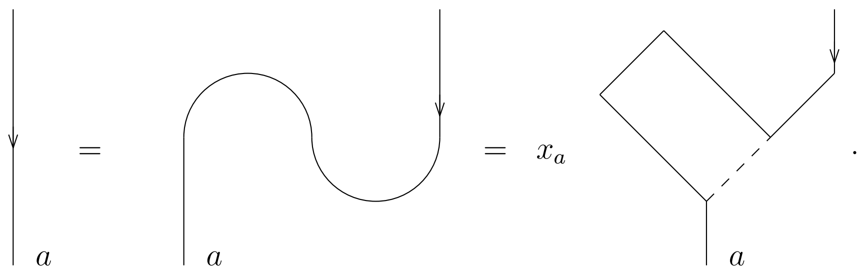
Rigidity yields the following lemma.

Lemma 2.3.3.

$$\left(F^{a^*}_{a^*aa^*}\right)_{\mathbb{1}\mathbb{1}} d_a = R^{\mathbb{1}}_{aa^*}\theta_a. \tag{2.3.55}$$

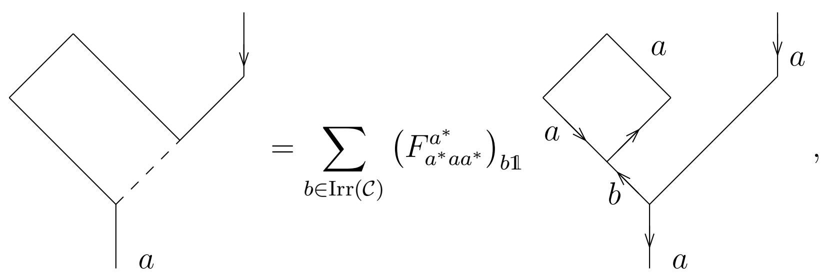
Proof. Again, let $x_a = \dfrac{d_a}{R^{\mathbb{1}}_{aa^*}\theta_a}$.

By rigidity,



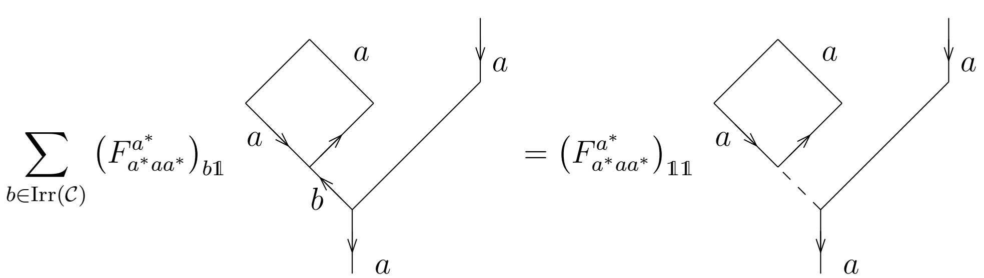
(2.3.56)

Since



(2.3.57)

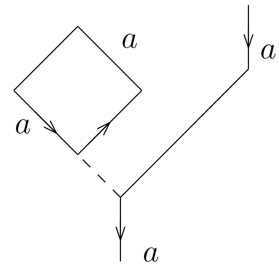
and the only summand on the right hand side that is possibly non-zero is given by the $b = \mathbb{1}$ term. In other words, we have



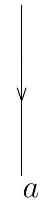
(2.3.58)

Putting the above equations together, we have

$= x_a \left(F^{a^*}_{a^* a a^*}\right)_{\mathbb{1}\mathbb{1}}$



$= x_a \left(F^{a^*}_{a^* a a^*}\right)_{\mathbb{1}\mathbb{1}}$



, (2.3.59)

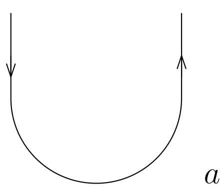
hence we have

$$x_a \left(F^{a^*}_{a^* a a^*}\right)_{\mathbb{1}\mathbb{1}} = 1. \tag{2.3.60}$$

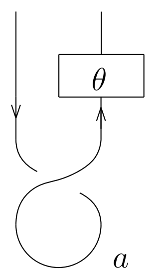
□

In particular, $\left(F^{a^*}_{a^* a a^*}\right)_{\mathbb{1}\mathbb{1}} \neq 0$.

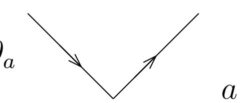
Similar to the normalization of ev_{a^*} (see Equation (2.3.47)), coev_{a^*} is given by



$=$

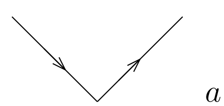


$= \dfrac{d_a}{R^{\mathbb{1}}_{aa^*}\theta_a} \cdot R^{\mathbb{1}}_{aa^*}\theta_a$



(2.3.61)

$= d_a$



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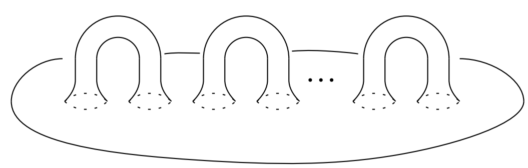
Chapter 3

TQFT and mapping class group representations

3.1 Cobordism categories

3.1.1 The category $\boldsymbol{Cob}_0^\bullet$

For each $g \in \mathbb{Z}_{\geq 0}$, we construct a compact, connected, oriented surface of genus g with exactly one boundary component which is homeomorphic to S^1 as follows. Pick a disc B^2 and cut out $2g$ open discs along a diameter of B^2. Along the cut boundary components glue in g cylinders such that the connected boundary components of each cylinder are glued to two consecutive holes with compatible orientation. For each g the above construction will be our model surface, and we will denote it by $\Sigma_{g,1}$.

$\Sigma_{g,1} =$  . (3.1.1)

For two surfaces $\Sigma_{g,1}$ and $\Sigma_{h,1}$, let

$$\Sigma_{[h,g]} := -\Sigma_{h,1} \cup_{S^1\times 0} S^1 \times [0,1] \cup_{S^1\times 1} \Sigma_{g,1} \tag{3.1.2}$$

be the closed oriented surface of genus $g+h$ obtained by gluing the original surfaces along their boundaries with a cylinder $S^1 \times [0,1]$ inserted. Moreover, it is assumed that the identifications $S^1 \times 0 \xrightarrow{\sim} (-\partial\Sigma_{h,1})$ and $S^1 \times 1 \xrightarrow{\sim} \partial\Sigma_{g,1}$ are orientation preserving diffeomorphisms. We give the following definition according to [Ker03].

Definition 3.1.1. *A* relative cobordism *from $\Sigma_{h,1}$ to $\Sigma_{g,1}$ is a pair (M, ϕ), where M is a compact oriented 3-manifold, and $\phi : \partial M \xrightarrow{\sim} \Sigma_{[h,g]}$ is a homeomorphism. Two relative cobordisms (M, ϕ) and (M', ϕ') are called* equivalent *if there is a homeomorphism $w : M \xrightarrow{\sim} M'$ such that the following diagram commutes*

$$\begin{array}{ccc} \partial M & \xrightarrow{\phi} & \Sigma_{[h,g]} \\ & {\scriptstyle w|_{\partial M}} \searrow \quad \nearrow {\scriptstyle \phi'} & \\ & \partial M' & \end{array} \quad . \tag{3.1.3}$$

We will write the equivalence class of (M, ϕ) as $[M, \phi]$.

The composition of two relative cobordisms $(M, \phi_M : \partial M \xrightarrow{\sim} \Sigma_{[h,g]})$ and $(N, \phi_N : \partial N \xrightarrow{\sim} \Sigma_{[g,k]})$ is the relative cobordism $(N \circ M, \phi_{N \circ M} : \partial N \circ M \to \Sigma_{[g,k]}$ obtained from gluing two cobordisms together along the common boundary piece $\Sigma_{h,1}$ using the maps ϕ_M and ϕ_N, and then shrinking the cylindrical part monotonously such that it has height 1. It is shown in [KL01] that the resulting homeomorphism class does not depend on the choices of M and N in their classes.

By the discussions above, we obtain a category whose objects are given by the set of standard surfaces $\Sigma_{g,1}$, and whose morphisms are the homeomorphism classes of relative cobordisms $[M, \phi]$. We denote this category by $\boldsymbol{Cob}_0^\bullet$.

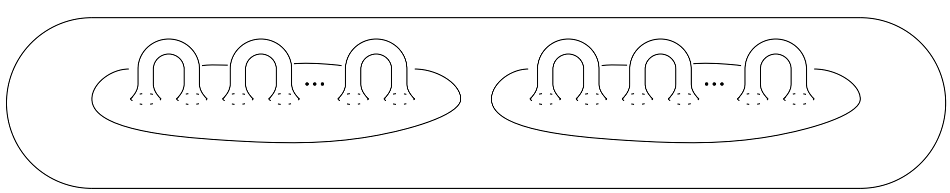
3.1.2 The category $\boldsymbol{Cob}^\bullet$

To construct topological quantum field theories, we will need more information encoded in the cobordisms than the topological structures given above. Namely, we are interested in *2-framed* cobordisms between surfaces in $\boldsymbol{Cob}_0^\bullet$. A 2-framed relative cobordism class can be characterized by a relative cobordism class $[M, \phi]$ together with a non-negative integer, which can be given by the signature of a 4-manifold bounding a particular closure of M. For a detailed definition, see [Ker99] and [Ker03].

The new category, with the same set of objects $\Sigma_{g,1}$ as in $\boldsymbol{Cob}_0^\bullet$, whose morphisms are 2-framed relative cobordisms classes, is denoted by $\boldsymbol{Cob}^\bullet$. We view it as an extension of $\boldsymbol{Cob}_0^\bullet$ by $\mathbb{Z}$ on the level of morphisms. In other words, there is an essentially surjective full functor

$$\boldsymbol{Cob}^\bullet \twoheadrightarrow \boldsymbol{Cob}_0^\bullet. \tag{3.1.4}$$

It is shown in [Ker03, KL01] that $\boldsymbol{Cob}_0^\bullet$ carries a monoidal category structure. We first describe the tensor product on objects. Fix a disc with two holes $P = D^2 - (D^2 \sqcup D^2)$. Given two surfaces $\Sigma_{g,1}$ and $\Sigma_{h,1}$, we sew them into the two holes of P such that $\Sigma_{g,1}$ is on the left and $\Sigma_{h,1}$ is on the right. Align the handles of $\Sigma_{g,1}$ and $\Sigma_{h,1}$ from left to right, then there is a unique homeomorphism $m_{g,h} : \Sigma_{g,1} \cup_{S^1} P \cup_{S^1} \Sigma_{h,1} \xrightarrow{\cong} \Sigma_{g+h,1}$ up to isotopy which maps the corresponding handles in order onto each other. Given two cobordisms $M : \Sigma_{g,1} \to \Sigma_{h,1}$ and $L : \Sigma_{p,1} \to \Sigma_{q,1}$, the tensor product is obtained by gluing the cobordisms into $P \times [0, 1]$ and using the $m_{g,h}$ and $m_{p,q}$ to adjust the boundary identifications so that $M \otimes N : \Sigma_{g+p,1} \to \Sigma_{h+q,1}$. It is shown in [KL01] that the tensor product defined above lifts to the 2-framing structure, therefore, endowing $\boldsymbol{Cob}^\bullet$ the structure of a monoidal category. Moreover, $\boldsymbol{Cob}^\bullet$ has a braiding. For details, see [KL01].



. (3.1.5)

Theorem 3.1.1 (Theorem 3, [Ker03])**.** $\boldsymbol{Cob}^\bullet$ is a braided monoidal category with objects $\Sigma_{g,1} = \Sigma_{1,1}^{\otimes g}$.

3.2 Topological quantum field theory

Let $\mathcal{C}$ be a modular category. Let $\mathcal{E} = \bigoplus_{a \in \mathrm{Irr}(\mathcal{C})} a^* \otimes a \in \mathrm{Ob}(\mathcal{C})$. The following definition is a special case of the extended TQFT formalism in [KL01].

Definition 3.2.1. *A* Topological Quantum Field Theory *for compact connected oriented surfaces with one boundary component (TQFT) associated to the modular category $\mathcal{C}$ is a monoidal functor*

$$\mathcal{V} : \boldsymbol{Cob}^{\bullet} \longrightarrow \mathcal{C} \tag{3.2.1}$$

such that $\mathcal{V}(\Sigma_{1,1}) = \mathcal{E}$.

Remark 3.2.1. Note that if $\mathcal{V}$ is a TQFT, then $\Sigma_{g,1} = \Sigma_{1,1}^{\otimes g}$ implies that for any $g \in \mathbb{Z}_{\geq 1}$,

$$\mathcal{V}(\Sigma_{g,1}) = \mathcal{V}(\Sigma_{1,1}^{\otimes g}) = (\mathcal{V}(\Sigma_{1,1}))^{\otimes g} = \mathcal{E}^{\otimes g}. \tag{3.2.2}$$

Definition 3.2.2. *Let $\mathcal{V} : \boldsymbol{Cob}^{\bullet} \longrightarrow \mathcal{C}$ be a TQFT. For any $a \in \mathrm{Ob}(\mathcal{C})$, a* special-ization *of $\mathcal{V}$ at a is the composition of functors*

$$\mathcal{V}^a : \boldsymbol{Cob}^{\bullet} \xrightarrow{\mathcal{V}} \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(a,\cdot)} \boldsymbol{Vec}, \tag{3.2.3}$$

*where **Vec** stands for the tensor category of vector spaces over $\mathbb{C}$.*

We introduce the following notation for simplicity. First of all, for $a \in \mathrm{Ob}(\mathcal{C})$, we denote the specialized spaces by

$$
\begin{aligned}
\mathbb{V}^a_{g,1} &:= \mathcal{V}^a(\Sigma_{g,1}) = \mathrm{Hom}_{\mathcal{C}}(a, \mathcal{E}^{\otimes g}) \\
&= \bigoplus_{(b_1,\ldots,b_g)\in \mathrm{Irr}(\mathcal{C})^g} \mathrm{Hom}_{\mathcal{C}}(a, b_1^* \otimes b_1 \otimes \cdots \otimes b_g^* \otimes b_g).
\end{aligned} \tag{3.2.4}
$$

3.3 Mapping class group representations

3.3.1 Generators and tangle presentations of $\widetilde{\Gamma_{g,1}}$

One of the most significant applications of TQFT is the implied existence of the extended mapping class group representations. Here we briefly recall the definition of the mapping class group.

Definition 3.3.1. *The* mapping class group *of $\Sigma_{g,1}$, denoted by $\Gamma_{g,1}$, is the group of isotopy classes of orientation preserving homeomorphisms of $\Sigma_{g,1}$ onto itself leaving the boundary component $\partial\Sigma_{g,1} \cong S^1$ pointwise fixed.*

By abuse of notation, we do not distinguish a representative orientation preserving homeomorphism and its class in $\Gamma_{g,1}$. To any $\gamma \in \Gamma_{g,1}$, we consider a cobordism denoted by C_γ. As a 3-manifold, C_γ is $\Sigma_{g,1} \times [0,1]$ together with boundary identification to $\Sigma_{[g,g]}$ given by $\mathrm{id}_{\Sigma_{g,1}}$ on $\Sigma_{g,1} \times 0$ and γ on $\Sigma_{g,1} \times 1$. The cobordism C_γ depends only on the isotopy class of $\gamma \in \Gamma_{g,1}$.

Let $\widetilde{\Gamma_{g,1}}$ be the central extension of $\Gamma_{g,1}$ by $\mathbb{Z}$, which carries the 2-framing information. In other words, we have the following short exact sequence

$$
1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma_{g,1}} \longrightarrow \Gamma_{g,1} \longrightarrow 1. \tag{3.3.1}
$$

Let $\mathrm{Aut}_{\boldsymbol{Cob}^\bullet}(\Sigma_{g,1})$ be the group of automorphisms of $\Sigma_{g,1}$ in $\boldsymbol{Cob}^\bullet$.

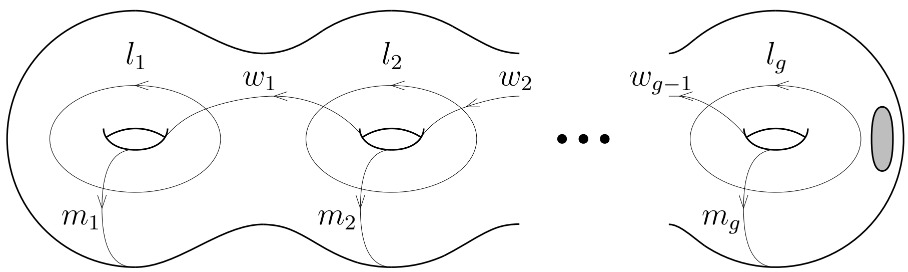
It is shown in [KL01] that

Theorem 3.3.1 ([KL01])**.** *For any $g \in \mathbb{Z}_{\geq 0}$, there is a group isomorphism*

$$\mathfrak{k}_g : \widetilde{\Gamma_{g,1}} \longrightarrow \mathrm{Aut}_{\boldsymbol{Cob}^\bullet}(\Sigma_{g,1}). \tag{3.3.2}$$

In particular, if we forget the 2-framing, for any $\gamma \in \Gamma_{g,1}$, $\mathfrak{k}_g$ sends γ to C_γ.

It is a standard result that $\Gamma_{g,1}$ can be generate by a finite set of Dehn twists along simple closed curves on $\Sigma_{g,1}$ (see, for example, [FM12]). We denote the generating curves by m_j, l_j and w_j (meridian, longitude and waist respectively), and the Dehn twists along the curves are denoted by M_j, L_j and W_j respectively.



(3.3.3)

To make the computations in the rest of the paper slightly easier, we use the set $\{T_j, \overline{S}_j, D_k\}$ as generators of the mapping class group, where

$$\begin{aligned} T_j &= M_j, \\ \overline{S}_j &= M_j^{-1} L_j^{-1} M_j^{-1}, \\ D_k &= M_k^{-1} M_{k+1}^{-1} W_k, \end{aligned} \tag{3.3.4}$$

for $j = 1, ..., g$, and $k = 1, ..., g-1$. We can lift the above generators of $\Gamma_{g,1}$ to be generators of $\widetilde{\Gamma_{g,1}}$. The way we lift the generators to $\widetilde{\Gamma_{g,1}}$ will be specified by the tangle presentation introduced in the next section. By an abuse of notation, we denote the lifts of the generators again by $T_j, \overline{S}_j, D_k$.

To generate $\widetilde{\Gamma_{g,1}}$, we need one more generator. Consider the cobordism $C_K \in \mathrm{Aut}_{\boldsymbol{Cob}^\bullet}(\Sigma_{g,1})$ given topologically by $\Sigma_{[g,g]} \times [0,1]$ with its framing changed by one. It is easy to see that the element

$$K := \mathfrak{k}_g^{-1}(C_K) \tag{3.3.5}$$

is a central element in $\widetilde{\Gamma_{g,1}}$ for any g. By [Ker03, Corollary 6] and Theorem 3.3.1 , we have

Proposition 3.3.1. The group $\widetilde{\Gamma_{g,1}}$ is generated by K, T_j, $\overline{S}_j$ for $j = 1, ..., g$, and D_k for $k = 1, ..., g-1$.

A useful way to study $\boldsymbol{Cob}^{\bullet}$ is to study its tangle presentations as shown in [Ker99, Ker03].

We first define the category $\boldsymbol{Tgl}$. Its objects are non-negative integers. To describe a morphism from k to l, we first define framed tangles. A *tangle* is an embedding of a compact one-dimensional manifold $\mathfrak{T}$ (not necessarily closed) into $\mathbb{R}^2 \times [0,1]$. A *framing* of a tangle $\mathfrak{T}$ is a trivialization of its normal bundle (see, for example, [MP94]).

For non-negative integers k, l, consider the set $\mathbf{Ftgl}_k^l$ of framed tangles in $\mathbb{R}^2 \times [0,1]$ with $2k$ end points labeled by $1^-, 1^+, ..., k^-, k^+$ at the bottom line $\mathbb{R}_x \times 0$, and with $2l$ end points similarly labeled at the top line $\mathbb{R}_x \times 1$, where $\mathbb{R}_x \subset \mathbb{R}^2$ is a fixed axis.

A framed tangle in $\mathbf{Ftgl}_k^l$ is called *admissible* if it has top, bottom, closed or through strands. A *top strand* is a component of the tangle that starts at $j^+ \in \mathbb{R}_x \times 1$ for some j and ends at the corresponding $j^- \in \mathbb{R}_x \times 1$. A *bottom strand* behaves similarly at the bottom line $\mathbb{R}_x \times 0$. A *closed strand* is a component homeomorphic to S^1 in the interior of $\mathbb{R}^2 \times [0,1]$. A *through strand* is a pair of components where one component starts at $j^+ \in \mathbb{R}_x \times 0$ and ends at either k^+ (or k^-) in $\mathbb{R}_x \times 1$, and the other component starts at $j^- \in \mathbb{R}_x \times 0$ and ends at k^- (or k^+ respectively) for some k and j.

We present an admissible framed tangle in a generic projection, subject to the second and third Reidemeister moves. We will assume that the framing are in the plane of projection. 2π-twists in the projections indicate the 2π-twist in the framing. Admissible framed tangles are subject to relations generalizing Kirby's calculus of links [Kir78]. We list these generalized Kirby relations following [Ker03]:

1. A Hopf link that is isolated from the rest of the diagram can be added or removed from the diagram if one component of the Hopf link is 0-framed and the other component is either 1- or 0-framed.

2. Any strand R_1 can be slid over a closed component R_2 by a 2-handle slide. For definition of a 2-handle slide, see [Kir78].

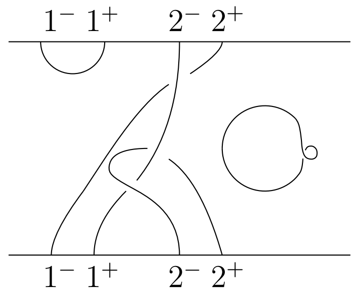
3. The boundary move, given by introducing two additional components in a vicinity of points $\{j^-, j^+\}$ at the bottom line. One is a bottom strand connecting j^- and j^+, the other is a closed strand going through the first arc, and finally, the outgoing strands are connected through the closed component.

For any two admissible framed tangles $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in $\mathbf{Ftgl}_k^l$, we write $\mathfrak{F}_1 \underset{GK}{\longleftrightarrow} \mathfrak{F}_2$ if we can get one admissible framed tangle from the other via a finite sequence of the three generalized Kirby moves given above. It is readily seen that $\underset{GK}{\longleftrightarrow}$ is an equivalence relation on the set of (admissible) framed tangles, and we call an equivalence class of admissible framed tangles with respect to $\underset{GK}{\longleftrightarrow}$ a *tangle class.*

Now we define Hom sets in $\boldsymbol{Tgl}$. For any $k, l \in \mathrm{Ob}(\boldsymbol{Tgl})$, the set of morphisms from k to l is the set of tangle classes from k to l. In other words, we have

$$\mathrm{Hom}_{\boldsymbol{Tgl}}(k, l) = \{\mathfrak{F} \in \mathbf{Ftgl}_k^l \mid \mathfrak{F} \text{ is admissible}\}/ \underset{GK}{\longleftrightarrow} . \qquad (3.3.6)$$

An example of (a representative in the equivalence class of) an admissible framed tangle projection is given by



$$\in \mathrm{Hom}_{\boldsymbol{Tgl}}(2,2). \tag{3.3.7}$$

Note that the isolated component (or closed strand) on the right is the projection of the embedding of S^1 with framing changed by -1.

It is shown in [Ker03, Ker99] that via surgery theory, there is an isomorphism of braided monoidal categories

$$\mathrm{Surg} : \boldsymbol{Tgl} \xrightarrow{\cong} \boldsymbol{Cob}^{\bullet}. \tag{3.3.8}$$

On the object level, this functor sends g to $\Sigma_{g,1}$. Together with the group isomorphism $\mathfrak{k}_g$ in Theorem 3.3.1 , we have a framed tangle presentation FT_γ for every element $\gamma \in \widetilde{\Gamma_{g,1}}$. More precisely, we have an isomorphism of groups given by the composition

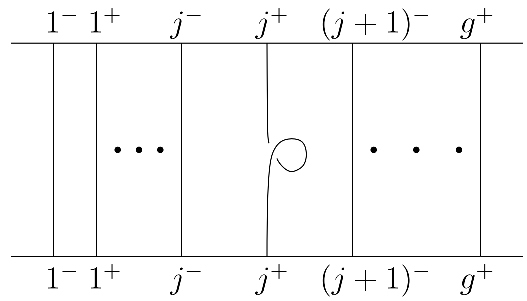
$$\begin{array}{ccccc} \widetilde{\Gamma_{g,1}} & \xrightarrow{\mathfrak{k}_g} & \mathrm{Aut}_{\boldsymbol{Cob}^{\bullet}}(\Sigma_{g,1}) & \xrightarrow{\mathrm{Surg}^{-1}} & \mathrm{Aut}_{\boldsymbol{Tgl}}(g) \\ \gamma & \longmapsto & C_\gamma & \longmapsto & FT_\gamma, \end{array} \tag{3.3.9}$$

where as usual, $\mathrm{Aut}_{\boldsymbol{Tgl}}(g)$ stands for the group of automorphisms of g in $\boldsymbol{Tgl}$.

Morphisms in $\boldsymbol{Tgl}$, or tangle classes, can be depicted by the generic projections (in other words, tangle diagrams) of their representatives, subject to the second and third Reidemeister moves and the usual moves for maxima and minima. Via the above isomorphism, we can give pictorial descriptions of generators of $\widetilde{\Gamma_{g,1}}$. We will call the tangle diagram of a representative of FT_γ the *tangle presentation* of γ.

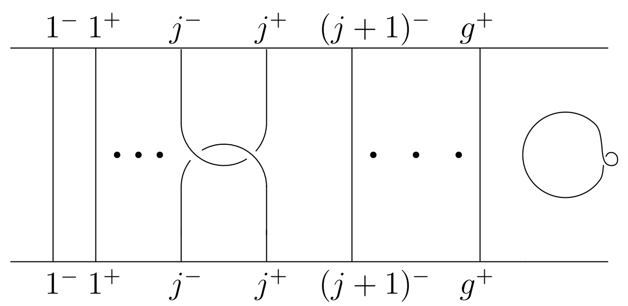
We give the list of tangle presentations of the generators of $\widetilde{\Gamma_{g,1}}$ following [Ker03].

The $\sim$ symbol below means the tangle presentation of the left hand side is given by the right hand side. Note that in [MP94], a similar result is given using a combinatorial method using Wajnryb's presentation of $\Gamma_{g,1}$ (see [Waj83]).

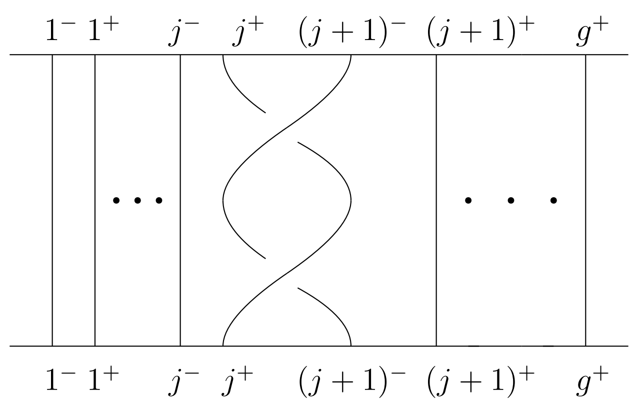
$T_j \sim$ 

, (3.3.10)

where the dots in the picture stands for vertical strands connecting k^- on the bottom line to k^- on the top line. Note that there is a 2π-twist in the framing of the j^+ strand. The tangle presentation of the other generators are given by

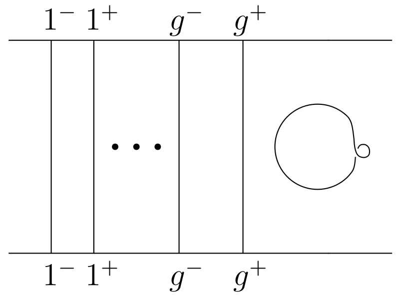
$S_j \sim$ 

, (3.3.11)

$D_j \sim$ 

, (3.3.12)

and

$K \sim$ 

. (3.3.13)

3.3.2 Mapping class group actions

Given a modular category $\mathcal{C}$ and a TQFT $\mathcal{V} : \boldsymbol{Cob}^{\bullet} \to \mathcal{C}$, its specialization $\mathcal{V}^a : \boldsymbol{Cob}^{\bullet} \to \boldsymbol{Vec}$ to any simple object $a \in \mathrm{Irr}(\mathcal{C})$ gives rise to representations of $\widetilde{\Gamma_{g,1}}$. To be more precise, for any $g \in \mathbb{Z}_{\geq 0}$, we have

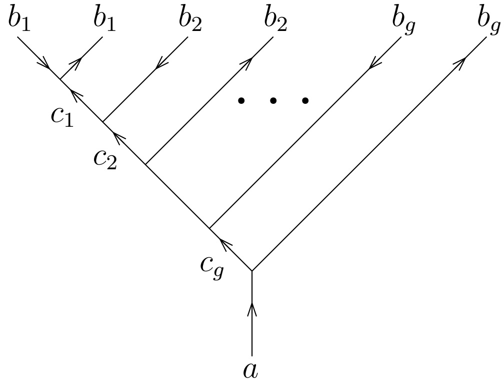
$$\rho^a_{g,1} : \widetilde{\Gamma_{g,1}} \xrightarrow{\mathfrak{t}_g} \mathrm{Aut}_{\boldsymbol{Cob}^{\bullet}}(\Sigma_{g,1}) \xrightarrow{\mathcal{V}^a} \mathrm{GL}(\mathcal{V}^a(\Sigma_{g,1})) = \mathrm{GL}(\mathbb{V}^a_{g,1}). \tag{3.3.14}$$

From now on, assume that $\mathcal{C}$ is multiplicity-free. We give the algorithm to explicitly compute the $\widetilde{\Gamma_{g,1}}$ action on $\mathbb{V}^a_{g,1}$ as follows.

Step 1. Fix a basis $\mathcal{B}$ for $\mathbb{V}^a_{g,1}$. Recall that

$$\mathbb{V}^a_{g,1} = \bigoplus_{(b_1,\dots,b_g)\in \mathrm{Irr}(\mathcal{C})^g} \mathrm{Hom}_{\mathcal{C}}(a, b_1^* \otimes b_1 \otimes \cdots \otimes b_g^* \otimes b_g), \tag{3.3.15}$$

and we can pick the basis for each of the direct summand space. For example, we can choose the left branched basis (cf. Equation (2.3.32)) in the form of

$\boldsymbol{v} =$ 

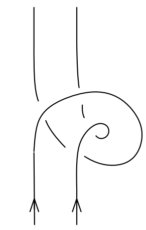
$$\in \mathrm{Hom}_{\mathcal{C}}(a, b_1^* \otimes b_1 \otimes \cdots b_g^* \otimes b_g). \quad (3.3.16)$$

Step 2. For any $\gamma \in \widetilde{\Gamma_{g,1}}$, stack its tangle presentation $FT_\gamma \in \mathrm{Aut}_{\boldsymbol{Tgl}}(g)$ on top of the basis vector $\boldsymbol{v}$ so that the bottom $2g$ end points of FT_γ coincide with the top $2g$ end points of $\boldsymbol{v}$. Delete the bottom and top lines of FT_γ as well as the labels of its end points. If a strand in FT_γ is connected to an edge of the basis element $\boldsymbol{v}$, it inherits the color and direction of the meeting edge. Let $\mathscr{L}_1, ..., \mathscr{L}_m$ be all the strands in FT_γ that are not connected to any edge of $\boldsymbol{v}$. For an m-tuple of simple objects

$$\vec{c} := (c_1, ..., c_m) \in \mathrm{Irr}(\mathcal{C})^m, \quad (3.3.17)$$

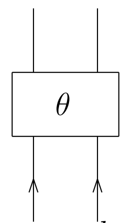
we color $\mathscr{L}_j$ by c_j for all $j = 1, ..., m$. Endow each $\mathscr{L}_j$ with a counter clockwise orientation. In this way, we get a colored and oriented framed tangle $FT_\gamma(\vec{c})$.

For every 2π-twist (inverse 2π-twist) on the colored strands (for example, the twists in the tangle presentations of $T_j, \overline{S}_j, K$ in last section), we replace the colored strands by the graphical calculus presentation of the ribbon structure θ (θ^{-1} respectively) on the objects indicated by the color (c.f. Definition 2.1.9) and Equation 2.3.13). For example, assume that for an m-tuple of simple objects $\vec{c}$, there is a 2π-twist on 2 strands colored by simple objects a and b in $FT_\gamma(\vec{c})$, then we have the following local picture



. (3.3.18)

We replace this local picture by



. (3.3.19)

After replacing all the (inverse) 2π-twists with the ribbon structure in $FT_\gamma(\vec{c})$, we denote the resulting picture by $\widetilde{FT}_\gamma(\vec{c})$.

Step 3. For each pair of basis element $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{B}$, let $\langle(\rho_{g,1}^a(\gamma))(\boldsymbol{v}), \boldsymbol{w}\rangle \in \mathbb{C}$ be the matrix coefficients of $\rho_{g,1}^a(\gamma)$ with respect to $\mathcal{B}$. In other words,

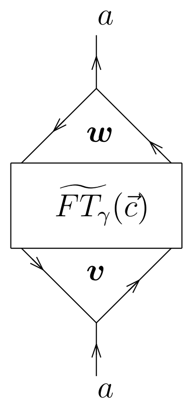
$$(\rho_{g,1}^a(\gamma))(\boldsymbol{v}) = \sum_{\boldsymbol{w}\in\mathcal{B}} \langle(\rho_{g,1}^a(\gamma))(\boldsymbol{v}), \boldsymbol{w}\rangle \boldsymbol{w} \tag{3.3.20}$$

As in Step 2, assume that FT_γ has m strands that are not connected to the edges of $\boldsymbol{v}$. For any $\vec{c} = (c_1, ..., c_m) \in \mathrm{Irr}(\mathcal{C})^m$, let

$$d_{\vec{c}} := \prod_{j=1}^{m} d_{c_j}. \tag{3.3.21}$$

We compute $\langle(\rho_{g,1}^a(\gamma))(\boldsymbol{v}), \boldsymbol{w}\rangle$ using the following formula:

$$\langle(\rho^a_{g,1}(\gamma))(\boldsymbol{v}),\boldsymbol{w}\rangle = \frac{1}{\mathcal{D}^m}\cdot\sum_{\vec{c}\in\mathrm{Irr}(\mathcal{C})^m} d_{\vec{c}}\cdot$$



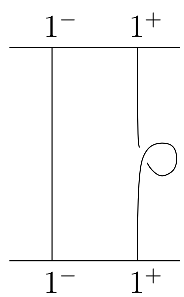
$$\in \mathrm{Hom}_{\mathcal{C}}(a,a)\cong\mathbb{C}. \qquad (3.3.22)$$

As an example, in the next section, we will compute $\rho^a_{1,1}$ following the above procedure.

3.3.3 Computation of $\rho^a_{1,1}$

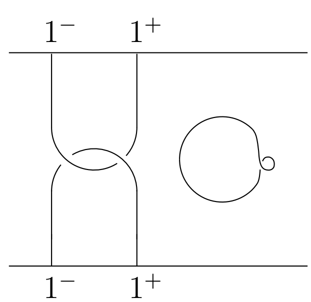
Let $a\in\mathrm{Irr}(\mathcal{C})$ be a simple object. To determine $\rho^a_{1,1}$, we only have to compute the action of the generators of $\widetilde{\Gamma_{1,1}}$ on the basis of $\mathbb{V}^a_{1,1}$. Note that $\widetilde{\Gamma_{1,1}}$ only has three generators T_1, $\overline{S}_1$ and K, whose tangle presentations are given by

$T_1 \sim$



, (3.3.23)

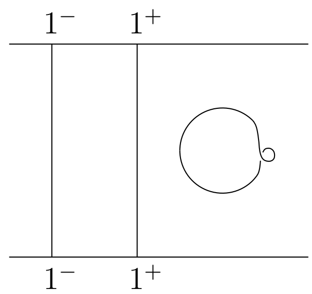
$\overline{S}_1 \sim$



(3.3.24)

and

$K \sim$



. (3.3.25)

For any $a \in \mathrm{Irr}(\mathcal{C})$, we write

$$T^a := \rho^a_{1,1}(T_1) : \mathbb{V}^a_{1,1} \longrightarrow \mathbb{V}^a_{1,1}, \tag{3.3.26}$$

$$\overline{S}^a := \rho^a_{1,1}(\overline{S}_1) : \mathbb{V}^a_{1,1} \longrightarrow \mathbb{V}^a_{1,1}, \tag{3.3.27}$$

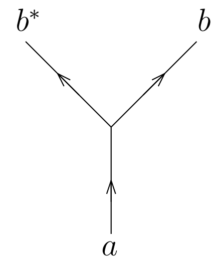
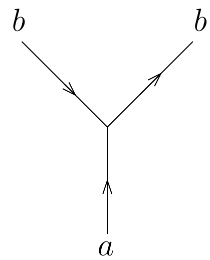
and

$$K^a := \rho^a_{1,1}(K) : \mathbb{V}^a_{1,1} \longrightarrow \mathbb{V}^a_{1,1}. \tag{3.3.28}$$

By definition,

$$\mathbb{V}^a_{1,1} = \bigoplus_{b \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(a, b^* \otimes b). \tag{3.3.29}$$

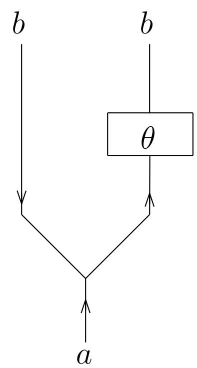
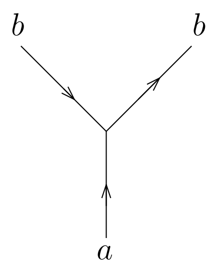
For each simple object b, we fix a basis vector for $\mathrm{Hom}_{\mathcal{C}}(a, b^* \otimes b)$, and denote the fixed basis vector by $v^a(b)$. We will call the basis $\{v^a(b)\}_{b \in \mathrm{Irr}(\mathcal{C})}$ the *standard basis* of $\mathbb{V}^a_{1,1}$. Pictorially, we write

$v^a(b) =$  $=$  . (3.3.30)

Theorem 3.3.2. *T^a acts diagonally on $\mathbb{V}^a_{1,1}$ by*

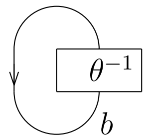
$$T^a(v^a(b)) = \theta_b \cdot v^a(b). \tag{3.3.31}$$

Proof. We compute the action of T^a by the algorithm given at the end of last section. Recall that once we stack the tangle presentation on the basis vector, we have to change the 2π-twist in the tangle diagram by the ribbon structure θ. By Equation (**??**),

$T^a(v^a(b)) =$  $= \theta_b$  $= \theta_b \cdot v^a(b).$ (3.3.32)

□

Next, consider

$$\overline{\xi}(\mathcal{C}) := \sum_{b\in\mathrm{Irr}(\mathcal{C})} \frac{1}{\mathcal{D}}$$ 

$$= \sum_{b\in\mathrm{Irr}(\mathcal{C})} \frac{\mathrm{tr}\left(d_b\theta_b^{-1}\right)}{\mathcal{D}} \qquad (3.3.33)$$

$$= \frac{1}{\mathcal{D}} \sum_{b\in\mathrm{Irr}(\mathcal{C})} d_b\theta_b^{-1}\mathrm{tr}\left(\mathrm{id}_b\right)$$

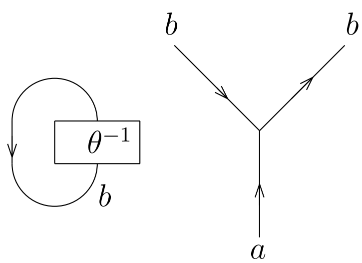
$$= \frac{1}{\mathcal{D}} \sum_{b\in\mathrm{Irr}(\mathcal{C})} d_b^2\theta_b^{-1}.$$

We will call $\overline{\xi}(\mathcal{C})$ the *inverse central charge* of $\mathcal{C}$.

Theorem 3.3.3.

$$K^a(v^a(b)) = \overline{\xi}(\mathcal{C})v^a(b). \qquad (3.3.34)$$

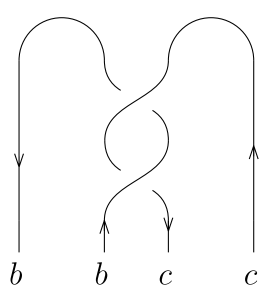
Proof.

$$K^a(v^a(b)) = \sum_{b\in\mathrm{Irr}(\mathcal{C})} \frac{1}{\mathcal{D}}$$  (3.3.35)

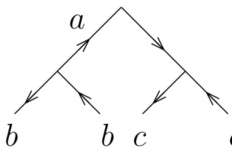
$$= \overline{\xi}(\mathcal{C})v^a(b).$$

□

To determine the action S^a, we first consider, for every pair of simple objects $b, c \in \mathrm{Irr}(\mathcal{C})$, the morphism

$$\Omega_{b,c} :=$$  $$\in \mathrm{Hom}_{\mathcal{C}}(b^* \otimes b \otimes c^* \otimes c, \mathbb{1}). \qquad (3.3.36)$$

Lemma 3.3.1.

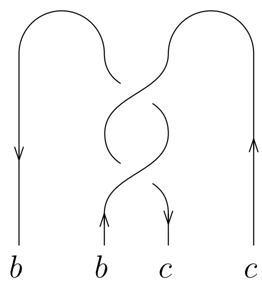
$$\Omega_{b,c} = \sum_{a \in \mathrm{Irr}(\mathcal{C})} \tilde{S}^a_{bc}$$  , (3.3.37)

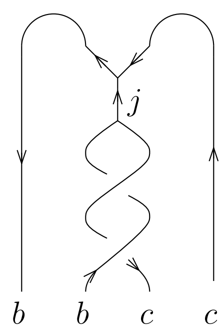
where

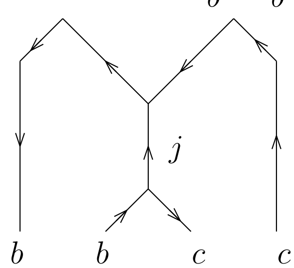
$$\tilde{S}^a_{bc} = \sum_{j \in \mathrm{Irr}(\mathcal{C})} R^j_{bc^*} R^j_{c^*b} \left(F^{c^*}_{b^*bc^*}\right)_{\mathbb{1}j} \left(G^{c^*}_{b^*bc^*}\right)_{ja} \left(F^{\mathbb{1}}_{ac^*c}\right)_{c^*a^*}. \tag{3.3.38}$$

Note that the summands on the right hand side form a basis for the space $\mathrm{Hom}_{\mathcal{C}}(b^* \otimes b \otimes c^* \otimes c, \mathbb{1})$.

Proof. By Lemma 2.3.1 and the normalization (2.3.43), we have

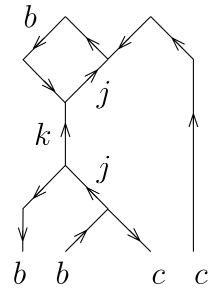


$$= \sum_{j \in \mathrm{Irr}(\mathcal{C})}$$  (3.3.39)

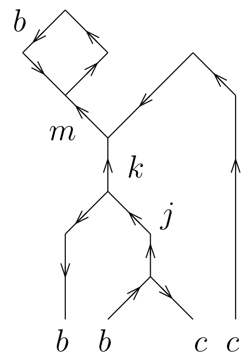
$$= \sum_{j \in \mathrm{Irr}(\mathcal{C})} R^j_{bc^*} R^j_{c^*b}$$  .

By Lemma 2.3.1 again, we have

$$\Omega_{b,c} = \sum_{j,\ k\in\mathrm{Irr}(\mathcal{C})} R^{j}_{bc^*} R^{j}_{c^*b}$$



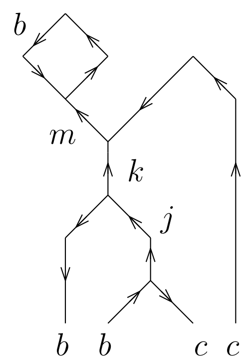
$$= \sum_{j,\ k\in\mathrm{Irr}(\mathcal{C})} R^{j}_{bc^*} R^{j}_{c^*b} \sum_{m\in\mathrm{Irr}(\mathcal{C})} \left(F^{k}_{b^*bc^*}\right)_{mj} \qquad (3.3.40)$$



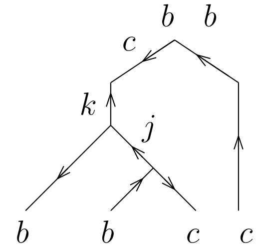
.

The picture summand on the right hand side of the previous equation is zero if $m \neq \mathbb{1}$. Hence, with the normalization (2.3.19), we have

$$\Omega_{b,c} = \sum_{j,\ k\in\mathrm{Irr}(\mathcal{C})} R^{j}_{bc^*} R^{j}_{c^*b} \sum_{m\in\mathrm{Irr}(\mathcal{C})} \left(F^{k}_{b^*bc^*}\right)_{mj}$$



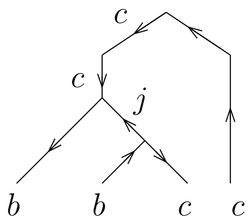
$$= \sum_{j,\ k\in\mathrm{Irr}(\mathcal{C})} R^{j}_{bc^*} R^{j}_{c^*b} \left(F^{k}_{b^*bc^*}\right)_{\mathbb{1}j} \qquad (3.3.41)$$



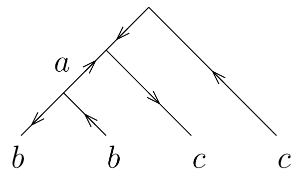
.

By Schur's Lemma, k has to be equal to c^* for the picture summand to be non-zero. Therefore, we have

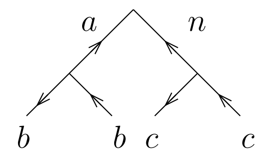
$$\Omega_{b,c} = \sum_{j \in \mathrm{Irr}(\mathcal{C})} R^{j}_{bc^*} R^{j}_{c^*b} \left(F^{c^*}_{b^*bc^*}\right)_{\mathbb{1}j}$$



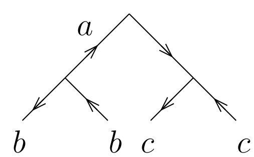
$$= \sum_{j \in \mathrm{Irr}(\mathcal{C})} R^{j}_{bc^*} R^{j}_{c^*b} \left(F^{c^*}_{b^*bc^*}\right)_{\mathbb{1}j} \sum_{a \in \mathrm{Irr}(\mathcal{C})} \left(G^{c^*}_{b^*bc^*}\right)_{ja}$$



$$= \sum_{j \in \mathrm{Irr}(\mathcal{C})} R^{j}_{bc^*} R^{j}_{c^*b} \left(F^{c^*}_{b^*bc^*}\right)_{\mathbb{1}j} \sum_{a \in \mathrm{Irr}(\mathcal{C})} \left(G^{c^*}_{b^*bc^*}\right)_{ja} \sum_{n \in \mathrm{Irr}(\mathcal{C})} \left(F^{\mathbb{1}}_{ac^*c}\right)_{c^*n}$$



$$= \sum_{a \in \mathrm{Irr}(\mathcal{C})} \sum_{j \in \mathrm{Irr}(\mathcal{C})} R^{j}_{bc^*} R^{j}_{c^*b} \left(F^{c^*}_{b^*bc^*}\right)_{\mathbb{1}j} \left(G^{c^*}_{b^*bc^*}\right)_{ja} \left(F^{\mathbb{1}}_{ac^*c}\right)_{c^*a^*}$$



. (3.3.42)

□

Now we are ready to compute the action of $\overline{S}^a$.

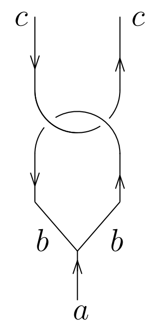
Theorem 3.3.4.

$$\overline{S}^a(v^a(b)) = \overline{\xi}(\mathcal{C}) \sum_{c \in \mathrm{Irr}(\mathcal{C})} S^a_{cb} \cdot v^a(c), \tag{3.3.43}$$

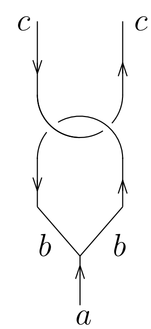
where

$$S^a_{cb} = \tilde{S}^a_{bc} \cdot \frac{d_c^2 \left(F^{c^*}_{c^*cc^*}\right)_{a^*\mathbb{1}} (G^{\mathbb{1}}_{a^*c^*c})_{c^*a} (F^a_{aa^*a})_{\mathbb{1}\mathbb{1}}}{\mathcal{D} \, (F^{c^*}_{c^*cc^*})_{\mathbb{1}\mathbb{1}}}. \tag{3.3.44}$$

Proof. By Equations (3.3.27) and (3.3.33), we have

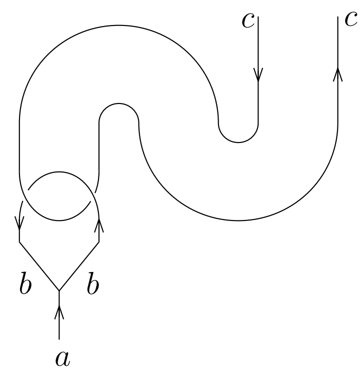
$$\overline{S}^a(v^a(b)) = \overline{\xi(\mathcal{C})} \cdot \sum_{c \in \mathrm{Irr}(\mathcal{C})} \frac{d_c}{\mathcal{D}} \cdot$$  $$. \tag{3.3.45}$$

Hence, it suffices to show that

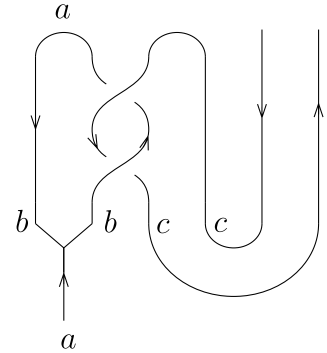
$$\mathcal{G}_{a,b} := \sum_{c \in \mathrm{Irr}(\mathcal{C})} \frac{d_c}{\mathcal{D}} \cdot$$  $$= \sum_{c \in \mathrm{Irr}(\mathcal{C})} S^a_{cb} \cdot v^a(c), \tag{3.3.46}$$

where S^a_{cb} is as in Equation (3.3.44).

By rigidity,

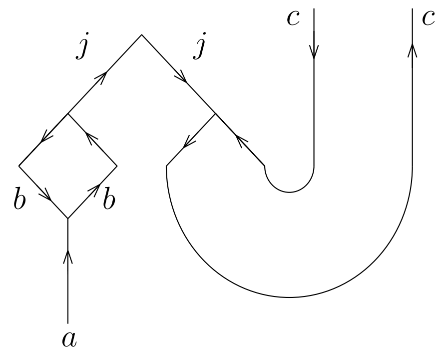
$$\mathcal{G}_{a,b} = \sum_{c \in \mathrm{Irr}(\mathcal{C})} \frac{d_c}{\mathcal{D}} \cdot$$ 

(3.3.47)

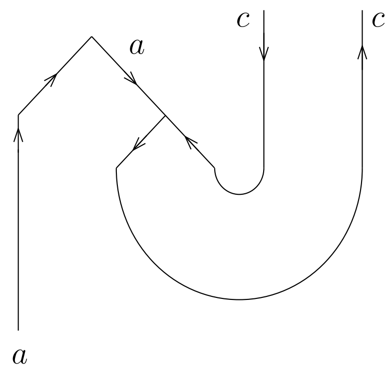
$$= \sum_{c \in \mathrm{Irr}(\mathcal{C})} \frac{d_c}{\mathcal{D}} \cdot$$  $$.$$

By Lemma 3.3.1 ,

$$\mathcal{G}_{a,b} = \sum_{c \in \mathrm{Irr}(\mathcal{C})} \sum_{j \in \mathrm{Irr}(\mathcal{C})} \tilde{S}^{j}_{bc} \cdot \frac{d_c}{\mathcal{D}} \cdot$$



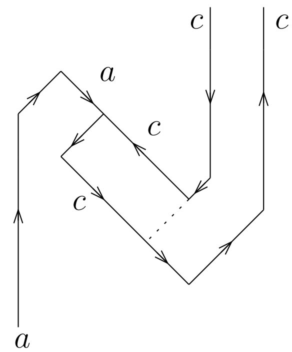
$$= \sum_{c \in \mathrm{Irr}(\mathcal{C})} \tilde{S}^{a}_{bc} \cdot \frac{d_c}{\mathcal{D}} \cdot \qquad (3.3.48)$$



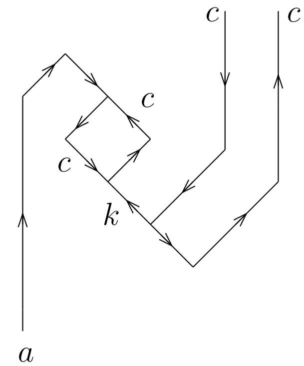
,

where the equality is given by the box-elimination normalization (2.3.19). By the normalization of coevaluations,

$$\mathcal{G}_{a,b} = \sum_{c \in \mathrm{Irr}(\mathcal{C})} \tilde{S}^a_{bc} \cdot \frac{d_c^2}{\mathcal{D}\left(F^{c^*}_{c^*cc^*}\right)_{\mathbb{1}\mathbb{1}}} \cdot$$

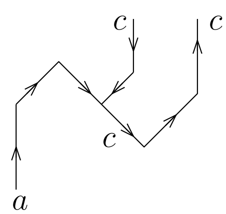


$$= \sum_{c \in \mathrm{Irr}(\mathcal{C})} \sum_{k \in \mathrm{Irr}(\mathcal{C})} \tilde{S}^a_{bc} \cdot \frac{d_c^2 \left(F^{c^*}_{c^*cc^*}\right)_{k\mathbb{1}}}{\mathcal{D}\left(F^{c^*}_{c^*cc^*}\right)_{\mathbb{1}\mathbb{1}}} \cdot \quad (3.3.49)$$

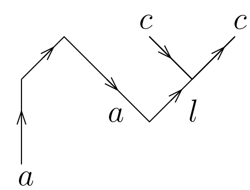


Again, k has to be equal to a^* by the box-elimination formula, which results in

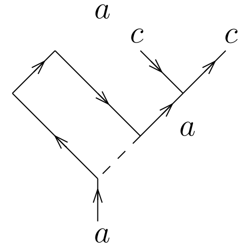
$$\mathcal{G}_{a,b} = \sum_{c \in \mathrm{Irr}(\mathcal{C})} \tilde{S}^a_{bc} \cdot \frac{d_c^2 \left(F^{c^*}_{c^*cc^*}\right)_{a^*\mathbb{1}}}{\mathcal{D}\left(F^{c^*}_{c^*cc^*}\right)_{\mathbb{1}\mathbb{1}}} \cdot$$



$$= \sum_{c \in \mathrm{Irr}(\mathcal{C})} \sum_{l \in \mathrm{Irr}(\mathcal{C})} \tilde{S}^a_{bc} \cdot \frac{d_c^2 \left(F^{c^*}_{c^*cc^*}\right)_{a^*\mathbb{1}} \left(G^{\mathbb{1}}_{a^*c^*c}\right)_{c^*l}}{\mathcal{D}\left(F^{c^*}_{c^*cc^*}\right)_{\mathbb{1}\mathbb{1}}} \cdot \quad (3.3.50)$$

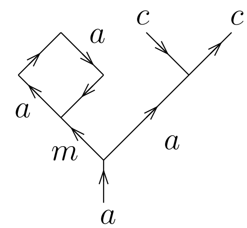


$$= \sum_{c \in \mathrm{Irr}(\mathcal{C})} \tilde{S}^a_{bc} \cdot \frac{d_c^2 \left(F^{c^*}_{c^*cc^*}\right)_{a^*\mathbb{1}} \left(G^{\mathbb{1}}_{a^*c^*c}\right)_{c^*a}}{\mathcal{D}\left(F^{c^*}_{c^*cc^*}\right)_{\mathbb{1}\mathbb{1}}} \cdot$$

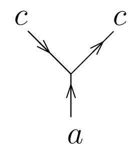


By Schur's Lemma, there is only one choice of l in the second sum that yields a non-zero term, namely, $l = a$. Therefore,

$$\mathcal{G}_{a,b} = \sum_{c\in \mathrm{Irr}(\mathcal{C})} \sum_{m\in \mathrm{Irr}(\mathcal{C})} \tilde{S}^a_{bc} \cdot \frac{d_c^2 \left(F^{c^*}_{c^*cc^*}\right)_{a^*\mathbb{1}} (G^{\mathbb{1}}_{a^*c^*c})_{c^*a} (F^a_{aa^*a})_{m\mathbb{1}}}{\mathcal{D}\left(F^{c^*}_{c^*cc^*}\right)_{\mathbb{1}\mathbb{1}}} \cdot$$



$$= \sum_{c\in \mathrm{Irr}(\mathcal{C})} \tilde{S}^a_{bc} \cdot \frac{d_c^2 \left(F^{c^*}_{c^*cc^*}\right)_{a^*\mathbb{1}} (G^{\mathbb{1}}_{a^*c^*c})_{c^*a} (F^a_{aa^*a})_{\mathbb{1}\mathbb{1}}}{\mathcal{D}\left(F^{c^*}_{c^*cc^*}\right)_{\mathbb{1}\mathbb{1}}} \cdot$$



$$= \sum_{c\in \mathrm{Irr}(\mathcal{C})} S^a_{cb} \cdot v^a(c), \tag{3.3.51}$$

where the first equality is again by the box-elimination normalization. □

3.3.4 Remarks on genus one mapping class groups

In this section, we take a closer look at mapping class groups in genus one. Let $\Gamma_{1,0}$ be the mapping class group of the torus (the genus one closed surface). It is well-known that $\Gamma_{1,0} \cong \mathrm{SL}(2,\mathbb{Z})$ [FM12]. We will identify $\Gamma_{1,1}$ as a central extension of $\Gamma_{1,0}$. We will also discuss the TQFT action of the Dehn twist on the boundary circle $\partial\Sigma_{1,1}$. The discussion in this section provides topological insights to the relationship between the $\widetilde{\Gamma_{1,1}}$ representations arising from the TQFT associated to a modular category $\mathcal{C}$ and the projective $\mathrm{SL}(2,\mathbb{Z})$ representation (Equation 2.2.17) given by $\mathcal{C}$, which will be discussed in more detail in Chapter 4 and Chapter 6. We will also have an algebraic description of the generators of $\widetilde{\Gamma_{1,1}}$ given in Section 3.3.3.

We adopt the following group presentation of $\mathrm{SL}(2,\mathbb{Z})$ in this section:

$$\mathrm{SL}(2,\mathbb{Z}) \cong \langle \omega_1, \omega_2 \mid \omega_1\omega_2\omega_1 = \omega_2\omega_1\omega_2,\ (\omega_1\omega_2)^6 = \mathrm{Id}_2 \rangle. \tag{3.3.52}$$

Note that we can write down ω_1 and ω_2 explicitly as

$$\omega_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{3.3.53}$$

Let $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$ be the universal central extension of $\mathrm{SL}(2,\mathbb{Z})$. It has the group presentation

$$\widetilde{\mathrm{SL}(2,\mathbb{Z})} \cong \langle \tilde{\omega}_1, \tilde{\omega}_2 \mid \tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_1 = \tilde{\omega}_2\tilde{\omega}_1\tilde{\omega}_2 \rangle. \tag{3.3.54}$$

The map $\widetilde{\mathrm{SL}(2,\mathbb{Z})} \longrightarrow \mathrm{SL}(2,\mathbb{Z})$ sending $\tilde{\omega}_j$ to ω_j for $j = 1, 2$ is a surjective homomorphism with kernel $\langle (\tilde{\omega}_1\tilde{\omega}_2)^6 \rangle \cong \mathbb{Z}$. It is easy to see from the relation in Equation (3.3.54) that $(\tilde{\omega}_1\tilde{\omega}_2)^6$ is central in $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$. Hence, we have a short exact sequence associated to a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{SL}(2,\mathbb{Z})} \longrightarrow \mathrm{SL}(2,\mathbb{Z}) \longrightarrow 1. \tag{3.3.55}$$

The leap from an algebraic to a topological point of view is triggered by the observation that the group presentation in Equation (3.3.54) is also a group presentation of B_3, the braid group on 3 strands. Let b_1, b_2 be the generators of B_3, the braid group relation is exactly $b_1b_2b_2 = b_2b_1b_2$. Therefore, we have a group isomorphism

$$\widetilde{\mathrm{SL}(2,\mathbb{Z})} \xrightarrow{\cong} B_3 \tag{3.3.56}$$

sending $\tilde{\omega}_j$ to b_j for $j = 1, 2$.

Now, we consider $\Gamma_{1,1}$. As discussed in Section 3.3.1 , $\Gamma_{1,1}$ is generated by M_1 and L_1, the Dehn twists along m_1 and l_1 respectively (see Equation (3.3.3)). Birman-Hilden [FM12, Theorem 9.2] showed that the map

$$\mathrm{BH} : B_3 \longrightarrow \Gamma_{1,1} \tag{3.3.57}$$

sending b_1 to M_1 and b_2 to L_1 is a group isomorphism. Therefore, we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathrm{SL}(2,\mathbb{Z})} & \longrightarrow & \mathrm{SL}(2,\mathbb{Z}) & \longrightarrow & 1 \\
 & & \Big\downarrow \cong & & \Big\downarrow \cong & & \Big\downarrow \cong & & \\
 & & & & B_3 & & & & \\
 & & & & \mathrm{BH}\Big\downarrow \cong & & & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma_{1,1} & \longrightarrow & \Gamma_{1,0} & \longrightarrow & 1
\end{array}, \tag{3.3.58}$$

where the bottom row

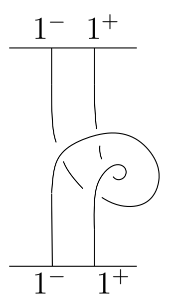
$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_{1,1} \longrightarrow \Gamma_{1,0} \longrightarrow 1 \tag{3.3.59}$$

identifies $\Gamma_{1,1}$ as a central extension of $\Gamma_{1,0}$. Moreover, Masbaum-Roberts [MR95] showed that the above extension of $\Gamma_{1,0}$ coincides with the framing central extension of $\Gamma_{1,0}$. According to our convention of notations, we have $\Gamma_{1,1} \cong \widetilde{\mathrm{SL}(2,\mathbb{Z})} \cong \widetilde{\Gamma_{1,0}}$.

In the light of the group isomorphism $\widetilde{\mathrm{SL}(2,\mathbb{Z})} \xrightarrow{\cong} B_3 \xrightarrow[\mathrm{BH}]{\cong} \Gamma_{1,1}$, we also view $\widetilde{\Gamma_{1,1}}$, the framing central extension of $\Gamma_{1,1}$, as a central extension of $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$.

We finish this chapter by a remark on the Dehn twist along the boundary curve of $\Sigma_{1,1}$, denoted by Q_1. The tangle presentation of Q_1 is given by [BK01]

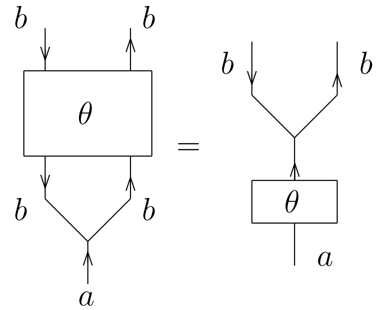
$Q_1 \sim$



. (3.3.60)

For $a \in \mathrm{Irr}(\mathcal{C})$, let $Q^a := \rho^a_{1,1}(Q_1)$. It is shown in [BK01, Section 3.1] that $Q^a = (\overline{S}^a)^4$. In principle, we can compute Q^a by the formulas in Theorem 3.3.4 . However, there is an easier way to compute Q^a by the naturality of the twist θ. We have, for any $v^a(b) \in \mathbb{V}^a_{1,1}$,

$$Q^a(v^a(b)) = \quad = \quad = \theta_a(v^a(b)). \tag{3.3.61}$$



As mentioned above, let $\widetilde{\Gamma_{1,0}}$ be the central extension of $\Gamma_{1,0}$ by 2-framing. It can also be generated by $\overline{S}_1, T_1$ and K. There is a surjective homomorphism $\mathcal{U} : \widetilde{\Gamma_{1,1}} \to \widetilde{\Gamma_{1,0}}$ mapping Q_1 to the identity element, and mapping every other generator to itself. In fact, $\mathcal{U}$ is a lifting of the map $\Gamma_{1,1} \longrightarrow \Gamma_{1,0}$ in Equation (3.3.59) to the central extensions.

Let $a \in \mathrm{Irr}(\mathcal{C})$ be a simple object in a modular category $\mathcal{C}$. If $\theta_a = 1$, the boundary Dehn twist acts as identity on $\mathbb{V}^a_{1,1}$. As a result, $\rho^a_{1,1}$ factors through a $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$ representation on $\mathbb{V}^a_{1,1}$. More precisely, if $\theta_a = 1$, there exists a group homomorphism $\mathcal{X}^a : \widetilde{\mathrm{SL}(2,\mathbb{Z})} \to \mathrm{GL}(\mathbb{V}^a_{1,1})$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\Gamma_{1,1}} & \xrightarrow{\quad\rho^a_{1,1}\quad} & \mathrm{GL}(\mathbb{V}^a_{1,1}) \\ & {\scriptstyle \mathcal{U}} \searrow \qquad \nearrow {\scriptstyle \mathcal{X}^a} & \\ & \widetilde{\Gamma_{1,0}} \cong \widetilde{\mathrm{SL}(2,\mathbb{Z})} & \end{array} \quad . \tag{3.3.62}$$

Chapter 4

The $\mathrm{SO}(p)_2$ modular categories

In this chapter, we give a detailed introduction to a family of the modular categories $\mathrm{SO}(p)_2$, whose associated TQFT will be the main focus of the rest of the paper. From now on, let $p = 2r + 1$ be an odd prime.

4.1 Introduction

The categories $\mathrm{SO}(p)_2$ come from quantum groups associated to $\mathfrak{so}(p)$ at roots of unity. For $q' = \exp\left(\frac{1}{2p}\pi i\right)$, we consider Lusztig's [Lus93] "modified form" of the Drinfeld-Jimbo quantum group $U_{q'}(\mathfrak{so}(p))$. Then we take the subcategory of tilting modules [AP95, APW95, And92] over $U_{q'}(\mathfrak{so}(p))$. The ribbon Hopf algebra structure of $U_{q'}(\mathfrak{so}(p))$ structure (comultiplication, universal R-matrix, antipode, ribbon element etc.) endows the abelian $\mathbb{C}$-linear category of tilting modules of $U_{q'}(\mathfrak{so}(p))$ with a ribbon structure. However, this category is neither semisimple nor finite, so to get a ribbon fusion category, we need to take the semisimple quotient of the category of the tilting modules [AP95] (in particular, we are able to get rid of objects whose quantum dimensions are 0).

In this way, we get a family of ribbon fusion categories $\mathrm{SO}(p)_2$, where the subscript 2, called the *level*, is a reminder of the order of q', which is $4p$ in this case. In practice however, it is more convenient to work with $q := \exp\left(\frac{1}{p}\pi i\right)$ and $\zeta := \exp\left(\frac{2}{p}\pi i\right)$ instead of q'.

As we will see in the this chapter, the S-matrix is invertible, so we have a family of modular categories. For a detailed description of the construction of quantum groups at roots of unity, we refer the readers to [BK01, Tur10]; we also recommend the survey articles [Row06, Saw06].

Modular categories having the same fusion rules as $\mathrm{SO}(m)_2$ for odd $m \geq 3$ are called *metaplectic modular categories.* In [AFT16], the authors classified metaplectic modular categories up to monoidal equivalence by solving the pentagon and hexagon equations for F- and R-matrices. In [AFT16], the categorical data (F- and R-matrices) are determined by families of parameters, and different families of parameters gives rise to monoidally inequivalent modular categories. We pick the family of parameters in [AFT16] such that the corresponding F- and R-matrices coincide with the categorical data provided by the quantum group construction in [NR11, HNW14].

The major motivation to consider $\mathrm{SO}(p)_2$, or more generally, metaplectic modular categories, is two-fold. From a mathematical point of view, these categories are *weakly integral*, meaning that the square of quantum dimensions of their simple objects are integers, and there exists a subcategory that has the same fusion rules as $Rep(D_{2p})$, the category of finite dimensional complex representations of the Dihedral group D_{2p}. The relationship between these two categories are worth studying. The category $\mathrm{SO}(p)_2$ also appears as the equivariantization [DGNO10] of the Tamabara-Yamagami categories [TY98]. Moreover, $\mathrm{SO}(p)_2$ is intimately related to the metaplectic link invariants of Goldschmidt-Jones [GJ89]. Using the relationship between $\mathrm{SO}(p)_2$ and the metaplectic link invariant construction, Rowell and Wenzl proved that the braid group representations arising from the TQFTs associated to $\mathrm{SO}(p)_2$ have finite image [RW17]. Their result provide evidences of the the *Property-F conjecture* [NR11], which posts that the braid group representations arising from a TQFT associated to a modular category $\mathcal{C}$ have finite image if and only if the $\mathcal{C}$ is weakly integral.

Birman-Hilden showed that a generalization of the homomorphism BH in Equation (3.3.57) embeds B_n, the braid group on n strands into the mapping class group $\Gamma_{2n+1,1}$ for any $n \geq 1$ [BH73, FM12]. Hence it is also interesting to understand more about mapping class group representation. For example, one may want to know if it is possible to generalize the Property-F conjecture further to predict the size of the image of the TQFT mapping class group representations. Integrality of these representations is a particularly intriguing question, and it is closely related to image finiteness.

In addition, metaplectic modular categories provide mathematical formalism to describe aspects of models of topological phases of matter occurring in condensed matter systems. For more details, the reader is referred to [HNW13, HNW14, CW15].

4.2 Categorical data for $\mathrm{SO}(p)_2$

In this section, we follow [NR11, HNW14, AFT16] to give the categorical data of $\mathrm{SO}(p)_2$ for $p = 2r + 1$ an odd prime.

4.2.1 Fusion rules

The $\mathrm{SO}(p)_2$ modular category has $(r + 4)$ simple objects, which we will label as

$$\mathrm{Irr}(\mathrm{SO}(p)_2) = \{\mathbb{1}, Z, Y_1, ..., Y_r, X, X'\}. \tag{4.2.1}$$

The fusion rules are given as follows:

$$
\begin{aligned}
&Z \otimes Z \cong \mathbb{1}, \\
&Z \otimes X \cong X', \\
&Z \otimes Y_j \cong Y_j, \quad \forall j = 1, \cdots, r, \\
&X \otimes X \cong \mathbb{1} \oplus \bigoplus_{j=1}^{r} Y_j, \\
&X \otimes X' \cong Z \oplus \bigoplus_{j=1}^{r} Y_j, \\
&X \otimes Y_j \cong X \oplus X', \quad \forall j = 1, \cdots, r, \\
&Y_j \otimes Y_j \cong \mathbb{1} \oplus Z \oplus Y_{\min\{2j,\ p-2j\}}, \quad \forall j = 1, \cdots, r, \\
&Y_j \otimes Y_k \cong Y_{|j-k|} \oplus Y_{\min\{j+k,\ p-j-k\}}, \quad \forall 1 \leq j\ , k \leq r, \text{ and } j \neq k.
\end{aligned} \tag{4.2.2}
$$

We define the function

$$
[\quad] : \{0, 1, ..., p\} \longrightarrow \{0, 1, ..., r\} \tag{4.2.3}
$$

by

$$
[x] = \begin{cases} x & \text{if } 0 \leq x \leq r, \\ p - x & \text{otherwise.} \end{cases} \tag{4.2.4}
$$

Then we can write the last two equations in the fusion rules as

$$
\begin{aligned}
&Y_j \otimes Y_j \cong \mathbb{1} \oplus Z \oplus Y_{[2j]}, \quad \forall j = 1, \cdots, r, \\
&Y_j \otimes Y_k \cong Y_{[j-k]} \oplus Y_{[j+k]}, \quad \forall 1 \leq j\ , k \leq r, \text{ and } j \neq k.
\end{aligned} \tag{4.2.5}
$$

Note that in particular, $\mathrm{SO}(p)_2$ is multiplicity-free and self-dual. The quantum dimensions of simple objects are given by

$$
\begin{aligned}
&d_{\mathbb{1}} = d_Z = 1; \\
&d_{Y_j} = 2,\ \forall j = 1, ..., r; \\
&d_X = d_{X'} = \sqrt{p}.
\end{aligned} \tag{4.2.6}
$$

Recall that $p = 2r + 1$. The square root of the global dimension of $\mathrm{SO}(p)_2$ is

$$
\mathcal{D} = \sqrt{1 + 1 + 2^2 \cdot (r) + 2 \cdot (\sqrt{p})^2} = \sqrt{4p} = 2\sqrt{p}. \tag{4.2.7}
$$

Note that $\mathcal{D} \in \mathbb{Z}[\zeta, i]$ is not a unit in the ring $\mathbb{Z}[\zeta, i]$. By Theorem 3.3.4, there is a $\mathcal{D}$ factor in the denominator of every entry of S^a for any $a \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$. As we will see in the computations in this chapter and Chapter 5, this denominator factor often makes an entry in S^a not in $\mathbb{Z}[\zeta, i]$.

4.2.2 F- and R-matrices

We will use the F-matrices and R-matrices of the $\mathrm{SO}(p)_2$ categories given in [AFT16]. In fact, [AFT16] classifies all braided fusion categories with the same fusion rules as $\mathrm{SO}(p)_2$, in terms of several parameters. Hence, to extract the F- and R-matrix data for computation, we have to specify our choice of the parameters. The reason we choose the following parameters is to get the same F- and R-matrices data given by the quantum group construction (cf. [NR11, HNW14]).

Let

$$
q := \exp\left(\frac{1}{p}\pi i\right), \tag{4.2.8}
$$

and

$$\zeta := \exp\left(\frac{2}{p}\pi i\right) = q^2. \tag{4.2.9}$$

The pair of integers parameterizing F-matrices (Section 4.2 in [AFT16]) is chosen to be $(1, \kappa_p)$, where

$$\kappa_p = \exp\left(\frac{p(p+1)}{2}\pi i\right) = \begin{cases} 1 & \text{if } p \equiv 3 \mod 4, \\ -1 & \text{if } p \equiv 1 \mod 4. \end{cases} \tag{4.2.10}$$

We emphasize here the following properties of the F-matrices of $\mathrm{SO}(p)_2$, which will be extensively used in next chapter:

Property 1. The entries in an F-matrix $\left(F^l_{abc}\right)_{ef}$ are indexed in the order of

$$(\mathbb{1}, Z, Y_1, ..., Y_r, X, X'), \tag{4.2.11}$$

if $\left(F^l_{abc}\right)_{ef} \neq 0$ according to the fusion rules;

Property 2. The F-matrices are real and unitary;

Property 3.

$$\left(F^l_{abc}\right)^t = F^c_{lab},\ \left(G^l_{abc}\right)^t = G^c_{lab} \tag{4.2.12}$$

for all $a, b, c, l \in \mathrm{Irr}\left(\mathrm{SO}(p)_2\right)$, where for any matrix M, M^t stands for the transpose of M;

Property 4. for simple objects $a, b, c, l \in \mathrm{Irr}\left(\mathrm{SO}(p)_2\right)$, $F^a_{bcl} = 1$ if one or more of a, b, c, l are equal to $\mathbb{1}$.

According to the fusion rules, $\mathrm{SO}(p)_2$ is multiplicity-free and self-dual, so we can rewrite the formulas for $\tilde{S}^a_{bc}$ and S^a_{cb} in Lemma 3.3.1 and Theorem 3.3.4 as in the following proposition.

Proposition 4.2.1. *For any simple objects $a, b, c \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$,*

$$\tilde{S}^a_{bc} = \sum_{j \in \mathrm{Irr}(\mathcal{C})} R^j_{bc} R^j_{cb} \left(F^c_{bbc}\right)_{\mathbb{1}j} \left(G^c_{bbc}\right)_{ja}, \tag{4.2.13}$$

and

$$S^a_{cb} = \tilde{S}^a_{bc} \cdot \frac{d_c^2 \left(F^c_{ccc}\right)_{a\mathbb{1}} \left(F^a_{aaa}\right)_{\mathbb{1}\mathbb{1}}}{\mathcal{D} \left(F^c_{ccc}\right)_{\mathbb{1}\mathbb{1}}}. \tag{4.2.14}$$

Proof. By Property 4 of the F-matrices, $(F^{\mathbb{1}}_{ac^*c})_{c^*a^*}$ and $(G^{\mathbb{1}}_{a^*c^*c})_{c^*a}$ are both equal to 1. □

Note that, by Equation (4.2.12), $\tilde{S}^a$ is symmetric. Indeed, $(F^c_{bbc})_{\mathbb{1}j} = (F^c_{cbb})_{j\mathbb{1}} = \left(F^b_{ccb}\right)_{\mathbb{1}j}$, and similarly for the G-matrix entry. In other words, we have $\tilde{S}^a_{bc} = \tilde{S}^a_{cb}$.

By [HNW14], for any simple object $a \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$, we have

$$\theta_a = \exp\left(2h_a \pi i\right), \tag{4.2.15}$$

where h_a, with the above choice of parameters, are given by (Section 4.3 in [AFT16])

$$\begin{aligned} &h_{\mathbb{1}} = 0, \quad h_Z = 1, \\ &h_{Y_k} = \frac{k(p-k)}{2p}, \\ &h_X = \frac{r}{8}, \quad h_{X'} = \frac{r+4}{8}. \end{aligned} \tag{4.2.16}$$

Let

$$\psi = \theta_X = \exp\left(\frac{2r}{8}\pi i\right) = \exp\left(\frac{r}{4}\pi i\right), \tag{4.2.17}$$

then

$$\theta_{X'} = \exp\left(\frac{2(r+4)}{8}\pi i\right) = -\exp\left(\frac{r}{4}\pi i\right) = -\psi. \tag{4.2.18}$$

Note that $\psi^2 = \exp\left(\frac{r}{2}\pi i\right) = i^r$.

We also have

$$\theta_{\mathbb{1}} = \theta_Z = 1 \tag{4.2.19}$$

and

$$\begin{aligned} \theta_j := \theta_{Y_j} &= \exp\left(2h_{Y_j}\pi i\right) = \exp\left(\frac{j(p-j)}{p}\pi i\right) \\ &= \exp\left(\left(j - \frac{j^2}{p}\right)\pi i\right) = \exp\left(\left(j^2 - \frac{j^2}{p}\right)\pi i\right) \\ &= \exp\left(\frac{(p-1)j^2}{p}\pi i\right) = \exp\left(\frac{2rj^2}{p}\pi i\right) \\ &= \zeta^{rj^2}. \end{aligned} \tag{4.2.20}$$

The R-matrices are given by

$$R^c_{ab} = L(a,b)\left(\sigma_1\right)^c_{ab}\exp\left(\left(h_c - h_a - h_b\right)\pi i\right) \tag{4.2.21}$$

for any simple objects $a, b, c \in \mathrm{Irr}\left(\mathrm{SO}(p)_2\right)$, where

$$L(a,b) = \begin{cases} -1 & \text{if } (a,b) \in \{X, X'\}^2, \\ 1 & \text{otherwise.} \end{cases} \tag{4.2.22}$$

For any triple of simple objects a, b, c, $\left(\sigma_1\right)^c_{ab}$ equals 1 or -1, and the exact values are given as follows. Firstly,

$$\left(\sigma_1\right)^c_{ab} = \left(\sigma_1\right)^a_{bc} \tag{4.2.23}$$

for any $a, b, c \in \operatorname{Irr}(\mathrm{SO}(p)_2)$. In addition, $(\sigma_1)^c_{ab} = 1$ except for the following special triples of simple objects:

- If r is odd and $(j \bmod 4) \in \{1, 2\}$, then

$$(\sigma_1)^X_{Y_j X} = (\sigma_1)^{X'}_{Y_j X'} = (\sigma_1)^{X'}_{Y_j X} = (\sigma_1)^X_{Y_j X'} = -1. \tag{4.2.24}$$

- If r is even and $(j \bmod 4) \in \{2, 3\}$, then

$$(\sigma_1)^X_{Y_j X} = (\sigma_1)^{X'}_{Y_j X'} = (\sigma_1)^{X'}_{Y_j X} = (\sigma_1)^X_{Y_j X'} = -1. \tag{4.2.25}$$

- For any $j, k \in \{1, ..., r\}$,

$$(\sigma_1)^{Y_{[j+k]}}_{Y_j Y_k} = (-1)^{jk}. \tag{4.2.26}$$

Remark 4.2.1. In [AFT16], the data denoted by $(\sigma_2)^c_{ab}$ for simple objects a, b, c is also used in the description of R-matrices. It turns out that $(\sigma_2)^c_{ab}$ equals to 1 for any simple object a, b, c for our choice of parameters that determine the R-matrices. Hence, we omit the $(\sigma_2)^c_{ab}$ from further discussions.

Lemma 4.2.1. *For any $a, b, c \in \operatorname{Irr}(\mathrm{SO}(p)_2)$, we have*

$$(\sigma_1)^c_{ab} (\sigma_1)^c_{ba} = 1 \tag{4.2.27}$$

Proof. We only have to prove the lemma for the special values of σ_1 listed above. Hence it suffices to consider the cases $a, b, c \in \{Y_1, ..., Y_r, X, X'\}$. Note also that $(\sigma_1)^c_{ab}$ equals to 1 or -1, so if $a = b$,

$$(\sigma_1)^c_{ab} (\sigma_1)^c_{ba} = ((\sigma_1)^c_{ab})^2 = 1. \tag{4.2.28}$$

Note that with the invariance of σ_1 under cyclic permutations of indices as in Equation (4.2.23) and the fact that the statement will be the same if we interchange a and b, we can establish the statement for the cases where two indices are the same. We are are left with the cases where all of the indices are different, namely

$$(a,b,c) \in \{(Y_j, X, X'), (Y_j, Y_k, Y_{[j+k]}) \mid 1 \leq j,k \leq r\}. \tag{4.2.29}$$

Case 1. $(a,b,c) = (Y_j, X, X')$ for some $j = 1, ..., r$.

According to Equation (4.2.24) and (4.2.25)

$$(\sigma_1)^{X'}_{Y_j X} (\sigma_1)^{X'}_{X Y_j} = (\sigma_1)^{X'}_{Y_j X} (\sigma_1)^{X}_{Y_j X'} = \left((\sigma_1)^{X'}_{Y_j X}\right)^2 = 1. \tag{4.2.30}$$

Case 2. $(a,b,c) = (Y_j, Y_k, Y_{[j+k]})$.

By Equation (4.2.26),

$$(\sigma_1)^{Y_{[j+k]}}_{Y_j Y_k} (\sigma_1)^{Y_{[j+k]}}_{Y_k Y_j} = \left((-1)^{jk}\right)^2 = 1. \tag{4.2.31}$$

□

Corollary 4.2.1. *For any triple of simple objects* $a, b, c \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$,

$$R^c_{ab} R^c_{ba} = \exp\left(2\left(h_c - h_a - h_b\right)\pi i\right). \tag{4.2.32}$$

Proof. Observe that $L(a,b)L(b,a) = L(a,b)^2 = 1$. □

4.2.3 Modular data

In this section, we compute the modular data (S- and T-matrices) of $\mathrm{SO}(p)_2$. We first introduce some notation. Let

$$\epsilon_p = \exp\left(\frac{r^2}{2}\pi i\right) = \begin{cases} 1, & p \equiv 1 \pmod 4 \\ i, & p \equiv 3 \pmod 4 \end{cases}. \tag{4.2.33}$$

Let $\left(\frac{*}{*}\right)_L$ be the quadratic residue symbol (or Legendre symbol).

Lemma 4.2.2.

$$\exp\left(\frac{-r}{2}\pi i\right) \cdot \epsilon_p \cdot \left(\frac{r}{p}\right)_L = 1. \tag{4.2.34}$$

Proof. The following values of the quadratic residue symbol are well-known [Lan94, Page 77-78]:

$$\left(\frac{-1}{p}\right)_L = \exp\left(\frac{p-1}{2}\pi i\right) = \exp\left(r\pi i\right), \tag{4.2.35}$$

$$\begin{aligned} \left(\frac{2}{p}\right)_L &= \exp\left(\frac{p^2-1}{8}\pi i\right) \\ &= \exp\left(\frac{2r(2r+2)}{8}\pi i\right) \\ &= \exp\left(\frac{r(r+1)}{2}\pi i\right). \end{aligned} \tag{4.2.36}$$

Recall that $p = 2r+1$, we have $-2r \equiv 1 \mod p$. By the multiplicity of the quadratic residue symbol, we have

$$\begin{aligned} \left(\frac{r}{p}\right)_L &= \left(\frac{-1}{p}\right)_L \cdot \left(\frac{2}{p}\right)_L^{-1} \\ &= \exp\left((r - \frac{r^2+r}{2})\pi i\right) \\ &= \exp\left(\frac{r-r^2}{2}\pi i\right). \end{aligned} \tag{4.2.37}$$

As a result, we have

$$
\begin{aligned}
&\exp\left(\frac{-r}{2}\pi i\right)\cdot\epsilon_p\cdot\left(\frac{r}{p}\right)_L \\
&=\exp\left(\frac{-r}{2}\pi i\right)\cdot\exp\left(\frac{r^2}{2}\pi i\right)\cdot\exp\left(\frac{r-r^2}{2}\pi i\right) \\
&=\exp\left(\frac{-r+r^2+r-r^2}{2}\pi i\right) \\
&=1.
\end{aligned}
\tag{4.2.38}
$$

□

Remark 4.2.2. Note that ϵ_p and the quadratic residue symbol appear in the quadratic Gauss sum formula [Lan94, Page 86-87]:

$$
\sum_{j=0}^{p-1}\zeta^{rj^2}=\epsilon_p\cdot\left(\frac{r}{p}\right)_L\cdot\sqrt{p}. \tag{4.2.39}
$$

In particular, $\sqrt{p}\in\mathbb{Z}[\zeta,i]$. We will often use this formula in the rest of this paper.

We are now ready to compute the modular data. It is shown in [BK01] that the S-matrix can be computed by the following formula:

$$
\mathfrak{s}_{ab}=\mathfrak{s}_{ba}=\frac{1}{\mathcal{D}}\cdot\theta_a^{-1}\cdot\theta_b^{-1}\cdot\sum_c\theta_c\cdot d_c. \tag{4.2.40}
$$

In particular,

$$
\mathfrak{s}_{\mathbb{1}a}=\frac{d_a}{\mathcal{D}}. \tag{4.2.41}
$$

The reader is referred to Equations (4.2.17), (4.2.18), (4.2.19) and (4.2.20) for the values of twists θ_a for $a\in\operatorname{Irr}(\mathrm{SO}(p)_2)$. We have

$$
\mathfrak{s}_{\mathbb{1}\mathbb{1}}=\mathfrak{s}_{\mathbb{1}Z}=\frac{1}{2\sqrt{p}}, \tag{4.2.42}
$$

$$\mathfrak{s}_{\mathbb{1}Y_j} = \frac{2}{2\sqrt{p}} = \frac{1}{\sqrt{p}}, \tag{4.2.43}$$

$$\mathfrak{s}_{\mathbb{1}X} = \frac{\sqrt{p}}{2\sqrt{p}} = \frac{1}{2}. \tag{4.2.44}$$

$$\mathfrak{s}_{ZZ} = \frac{1}{\mathcal{D}} \cdot \theta_Z^{-2} \cdot \theta_{\mathbb{1}} \cdot d_{\mathbb{1}} = \frac{1}{2\sqrt{p}}, \tag{4.2.45}$$

$$\begin{aligned} \mathfrak{s}_{ZY_j} &= \frac{1}{\mathcal{D}} \cdot \theta_Z^{-1} \cdot \theta_{Y_j}^{-1} \cdot \theta_{Y_j} \cdot d_{Y_j} \\ &= \frac{1}{\mathcal{D}} \cdot 2 = \frac{1}{\sqrt{p}}, \quad j = 1, ..., r, \end{aligned} \tag{4.2.46}$$

$$\begin{aligned} \mathfrak{s}_{ZX} &= \frac{1}{\mathcal{D}} \cdot \theta_Z^{-1} \cdot \theta_X^{-1} \cdot \theta_{X'} \cdot d_{X'} \\ &= \frac{1}{\mathcal{D}} \cdot \theta_Z^{-1} \cdot \theta_X^{-1} \cdot (-\theta_X) \cdot d_{X'} \\ &= \ -\frac{\sqrt{p}}{2\sqrt{p}} = -\frac{1}{2}, \end{aligned} \tag{4.2.47}$$

$$\mathfrak{s}_{ZX'} = \frac{1}{\mathcal{D}} \cdot \theta_Z^{-1} \cdot \theta_{X'}^{-1} \cdot \theta_X \cdot d_X = -\frac{1}{2}. \tag{4.2.48}$$

$$\begin{aligned} \mathfrak{s}_{Y_jY_j} &= \frac{1}{\mathcal{D}} \cdot \theta_{Y_j}^{-2} \cdot \Big(\theta_{\mathbb{1}} \cdot d_{\mathbb{1}} + \theta_Z \cdot d_Z + \theta_{Y_{[2j]}} \cdot d_{Y_{[2j]}}\Big) \\ &= \frac{1}{\mathcal{D}} \cdot \zeta^{-2rj^2} \cdot \Big(1 + 1 + 2\zeta^{r\cdot(2j)^2}\Big) \\ &= \frac{2}{\mathcal{D}} \cdot \zeta^{-2rj^2} \cdot \Big(1 + \zeta^{4rj^2}\Big) \\ &= \frac{2}{2\sqrt{p}} \cdot \Big(\zeta^{-2rj^2} + \zeta^{2rj^2}\Big) \\ &= \frac{1}{\sqrt{p}} \cdot \Big(\zeta^{j^2} + \zeta^{-j^2}\Big), \quad j = 1, ..., r, \end{aligned} \tag{4.2.49}$$

where the last equality is based on the fact that $\zeta^p = \zeta^{2r+1} = 1$.

We also have, for any $j = 1, ..., r$,

$$
\begin{aligned}
\mathfrak{s}_{Y_j Y_k} &= \frac{1}{\mathcal{D}} \cdot \theta_{Y_j}^2 \cdot \left(\theta_{Y_{[j+k]}} \cdot d_{Y_{[j+k]}} + \theta_{Y_{[j-k]}} \cdot d_{Y_{[j-k]}} \right) \\
&= \frac{2}{\mathcal{D}} \cdot \zeta^{-r(j^2+k^2)} \cdot \left(\zeta^{r(j+k)^2} + \zeta^{r(j-k)^2} \right) \\
&= \frac{2}{\mathcal{D}} \cdot \left(\zeta^{r(j+k)^2 - rj^2 - rk^2} + \zeta^{r(j-k)^2 - rj^2 - rk^2} \right) \\
&= \frac{2}{2\sqrt{p}} \cdot \left(\zeta^{2rjk} + \zeta^{-2rjk} \right) \\
&= \frac{1}{\sqrt{p}} \cdot \left(\zeta^{jk} + \zeta^{-jk} \right), \quad 1 \le j, k \le r,
\end{aligned}
\tag{4.2.50}
$$

$$
\begin{aligned}
\mathfrak{s}_{Y_j X} &= \frac{1}{\mathcal{D}} \cdot \theta_{Y_j}^{-1} \cdot \theta_X^{-1} \cdot (\theta_X \cdot d_X + \theta_{X'} \cdot d_{X'}) \\
&= \frac{1}{\mathcal{D}} \cdot \theta_{Y_j}^{-1} \cdot \theta_X^{-1} \cdot (\theta_X \cdot d_X - \theta_X \cdot d_X) \\
&= 0,
\end{aligned}
\tag{4.2.51}
$$

$$
\begin{aligned}
\mathfrak{s}_{Y_j X'} &= \frac{1}{\mathcal{D}} \cdot \theta_{Y_j}^{-1} \cdot \theta_{X'}^{-1} \cdot (\theta_X \cdot d_X + \theta_{X'} \cdot d_{X'}) \\
&= 0.
\end{aligned}
\tag{4.2.52}
$$

$$
\begin{aligned}
\mathfrak{s}_{XX} &= \frac{1}{\mathcal{D}} \cdot \theta_X^{-2} \cdot \left(\theta_{\mathbb{1}} \cdot d_{\mathbb{1}} + \sum_{j=1}^{r} \theta_{Y_j} \cdot d_{Y_j} \right) \\
&= \frac{1}{\mathcal{D}} \cdot \exp\left(\frac{-r}{2} \pi i \right) \cdot \left(1 + \sum_{j=1}^{r} 2\zeta^{rj^2} \right) \\
&= \frac{1}{\mathcal{D}} \cdot \exp\left(\frac{-r}{2} \pi i \right) \cdot \sum_{j=0}^{p-1} \zeta^{rj^2} \\
&= \frac{1}{2\sqrt{p}} \cdot \exp\left(\frac{-r}{2} \pi i \right) \cdot \epsilon_p \cdot \left(\frac{r}{p} \right)_L \cdot \sqrt{p} \\
&= \frac{1}{2},
\end{aligned}
\tag{4.2.53}
$$

where the second last equation is the Gauss sum formula (4.2.39) and the last equation is given by Lemma 4.2.2 .

Based on the above computation, we have

$$\begin{aligned}\mathfrak{s}_{XX'} &= \frac{1}{\mathcal{D}} \cdot \theta_X^{-1} \cdot \theta_{X'}^{-1} \cdot \left(\theta_Z \cdot d_Z + \sum_{j=1}^{r} \theta_{Y_j} \cdot d_{Y_j}\right) \\ &= -\frac{1}{\mathcal{D}} \cdot \exp\left(\frac{-r}{2}\pi i\right) \cdot \left(1 + \sum_{j=1}^{r} 2\zeta^{rj^2}\right) \\ &= -\frac{1}{2}\end{aligned} \tag{4.2.54}$$

and

$$\begin{aligned}\mathfrak{s}_{X'X'} &= \frac{1}{\mathcal{D}} \cdot \theta_{X'}^{-2} \cdot \left(\theta_{\mathbb{1}} \cdot d_{\mathbb{1}} + \sum_{j=1}^{r} \theta_{Y_j} \cdot d_{Y_j}\right) \\ &= \frac{1}{\mathcal{D}} \cdot \exp\left(\frac{-r}{2}\pi i\right) \cdot \left(1 + \sum_{j=1}^{r} 2\zeta^{rj^2}\right) \\ &= \frac{1}{2}.\end{aligned} \tag{4.2.55}$$

We summarize the above computations in the following proposition.

Proposition 4.2.2.

$$\mathfrak{s} = \begin{pmatrix} \frac{1}{2\sqrt{p}} & \frac{1}{2\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot \boldsymbol{a}^t & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2\sqrt{p}} & \frac{1}{2\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot \boldsymbol{a}^t & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{p}} \cdot \boldsymbol{a} & \frac{1}{\sqrt{p}} \cdot \boldsymbol{a} & A & \mathbf{0} & \mathbf{0} \\ \frac{1}{2} & -\frac{1}{2} & \mathbf{0} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \mathbf{0} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \tag{4.2.56}$$

where

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{4.2.57}$$

is an $r \times 1$*-matrix,* A *is the* $r \times r$*-matrix with entries given by*

$$A_{jk} = \frac{2}{\sqrt{p}} \cdot \cos\left(\frac{2\pi jk}{p}\right) = \frac{1}{\sqrt{p}}\left(\zeta^{jk} + \zeta^{-jk}\right), \tag{4.2.58}$$

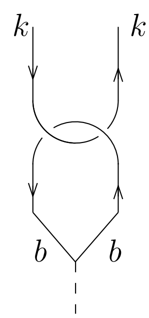
and $\mathbf{0}$ *is the zero matrix of suitable size.*

Proposition 4.2.3. *For any simple object* $b, c \in \mathrm{Irr}\left(\mathrm{SO}(p)_2\right)$*,*

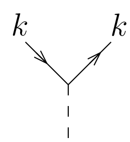
$$\mathfrak{s}_{cb} = d_c^{-1} S^{\mathbb{1}}_{cb} d_b. \tag{4.2.59}$$

Proof. Setting $a = \mathbb{1}$ in Theorem 3.3.4 , we have

$$\sum_{k \in \mathrm{Irr}(\mathcal{C})} \frac{d_k}{\mathcal{D}}$$



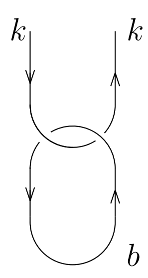
$$= \sum_{k \in \mathrm{Irr}(\mathcal{C})} S^{\mathbb{1}}_{kb}$$



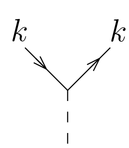
. (4.2.60)

By Equation (2.3.61), we have

$$\sum_{k \in \mathrm{Irr}(\mathcal{C})} \frac{d_k}{d_b \mathcal{D}}$$



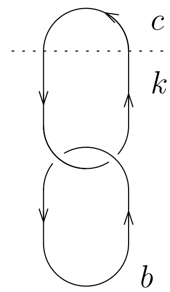
$$= \sum_{k \in \mathrm{Irr}(\mathcal{C})} S^{\mathbb{1}}_{kb}$$



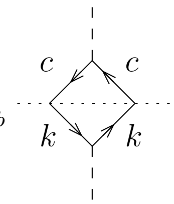
. (4.2.61)

By Equation (2.3.43), for any $c \in \mathrm{Irr}\left(\mathrm{SO}(p)_2\right)$ we have

$$\sum_{k\in\mathrm{Irr}(\mathcal{C})} \frac{d_k}{d_b\mathcal{D}}$$



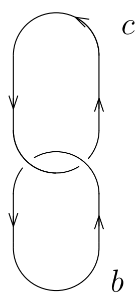
$$= \sum_{k\in\mathrm{Irr}(\mathcal{C})} S^{\mathbb{1}}_{kb}$$



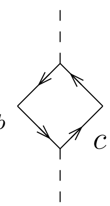
, (4.2.62)

where the dotted lines are meant to indicate where to stack ev_c. By Schur's Lemma, we can see that, the pictures summands on both sides are 0 if $k \neq c$. Together with the box-elimination, the above equation becomes

$$\frac{d_c}{d_b\mathcal{D}}$$



$$= S^{\mathbb{1}}_{cb}$$



$$= S^{\mathbb{1}}_{cb}. \tag{4.2.63}$$

By Equation (2.3.50) and the definition of $\mathfrak{s}$ (Equation (2.2.13)), we have

$$\frac{d_c\tilde{\mathfrak{s}}_{cb}}{d_b\mathcal{D}} = \frac{d_c\mathfrak{s}_{cb}}{d_b} = S^{\mathbb{1}}_{cb}, \tag{4.2.64}$$

and we are done by multiplying both sides by $d_c^{-1}d_b$. □

Note that by the proposition, $\mathfrak{s}$ and $S^{\mathbb{1}}$ are conjugate by a diagonal matrix.

Next, we compute $T^{\mathbb{1}}$.

By Theorem 3.3.2 we have the following proposition.

Proposition 4.2.4. *The matrix presentation of the meridian twist operator $T^{\mathbb{1}}$ with respect to the standard basis for $\mathbb{V}^{\mathbb{1}}_{1,1}$ is*

$$T^{\mathbb{1}} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \theta_1 & & & & \\ & & & \ddots & & & \\ & & & & \theta_r & & \\ & & & & & \psi & \\ & & & & & & -\psi \end{pmatrix}. \tag{4.2.65}$$

Note that by definition, $T^{\mathbb{1}}$ is equal to the T-matrix t.

Chapter 5

Integrality

In this chapter, we focus on the algebraic properties of the TQFT associated to $\mathrm{SO}(p)_2$. We prove the Main Theorem by finding, for each simple object $a \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$, a full-rank free lattice $\Lambda_a \subset \mathbb{V}^a_{1,1}$ over $\mathbb{Z}[\zeta, i]$ that is preserved by the $\widetilde{\Gamma_{1,1}}$-action. For simplicity, we will use $\mathbb{Z}[\zeta, i]$ and $\mathcal{O}$ interchangeably. We will call a number $x \in \mathbb{C}$ *integral* if $x \in \mathcal{O}$. A matrix M is called integral if all of its entries are integral, denoted by $M \sqsubset \mathcal{O}$.

Note it is sufficient to find a basis for $\mathbb{V}^a_{1,1}$, with respect to which, $\overline{S}^a$, T^a and K^a, are integral. We will call such a basis an *integral basis* of $\mathbb{V}^a_{1,1}$. It is readily seen that, once we have an integral basis $\mathcal{B}^a$ for $\mathbb{V}^a_{1,1}$, a $\widetilde{\Gamma_{1,1}}$-invariant full-rank free lattice in the statement of the Main Theorem can be given as $\Lambda_a := \mathrm{span}_{\mathcal{O}}(\mathcal{B}^a)$.

A closer look at the fusion rules eliminates the cases when $a \in \{X, X'\}$. More precisely, let

$$n_a := \dim(\mathbb{V}^a_{1,1}). \tag{5.0.1}$$

Then we have the following lemma.

Lemma 5.0.1.

$$\begin{aligned} &n_{\mathbb{1}} = r + 4 \\ &n_Z = r \\ &n_{Y_j} = 3,\ \forall j = 1, ..., r \\ &n_X = n_{X'} = 0 \end{aligned} \tag{5.0.2}$$

Proof. By definition, for any $a \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$,

$$n_a = \sum_{b \in \mathrm{Irr}(\mathcal{C})} N^a_{b^* b}, \tag{5.0.3}$$

where $N^a_{b^*b}$ are given by the fusion rules (4.2.2). □

As a result, we only have to consider the specialized spaces $\mathbb{V}^{\mathbb{1}}_{1,1}, \mathbb{V}^Z_{1,1}$ and $\mathbb{V}^{Y_k}_{1,1}$ for $k = 1, ..., r$.

Lemma 5.0.2.

$$\overline{\xi}(\mathrm{SO}(p)_2) = i^{3r}. \tag{5.0.4}$$

Proof. By definition,

$$
\begin{aligned}
\overline{\xi}(\mathcal{C}) &= \frac{1}{\mathcal{D}} \sum_{a \in \mathrm{Irr}(\mathcal{C})} d_a^2 \theta_a^{-1} \\
&= \frac{1}{2\sqrt{p}} \left(1 + 1 + 4 \sum_{j=1}^{r} \zeta^{-rj^2} + p\psi^{-1} + p(-\psi)^{-1} \right) \\
&= \frac{1}{2\sqrt{p}} \cdot 2 \sum_{j=0}^{p-1} \zeta^{-rj^2} \\
&= \frac{1}{2\sqrt{p}} \cdot 2 \cdot \epsilon_p \cdot \left(\frac{-r}{p} \right)_L \cdot \sqrt{p} \\
&= \left(\frac{-1}{p} \right)_L \cdot \epsilon_p \cdot \left(\frac{r}{p} \right)_L \\
&= \exp\left(\left(r + \frac{r}{2} \right) \pi i \right) \\
&= \exp\left(\frac{3r}{2} \pi i \right) = i^{3r}
\end{aligned}
\tag{5.0.5}
$$

by the quadratic Gauss sum formula (4.2.39) and Lemma 4.2.2 . ☐

In particular, $\overline{\xi}(\mathcal{C}) \in \mathcal{O}$. Hence, by Theorem 3.3.3 , $K^a = i^{3r} \cdot \mathrm{Id}_{n_a} \sqsubset \mathcal{O}$ for all $a \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$, where n_a is as in Equation (5.0.1), and Id_{n_a} is the identity matrix of size n_a. Moreover, as K^a is central, it is integral under any change of basis. Therefore, it suffices to find a basis for $\mathbb{V}^a_{1,1}$ with respect to which $\overline{S}^a \sqsubset \mathcal{O}$ and $T^a \sqsubset \mathcal{O}$.

We prove integrality via case-by-case study of the specialized spaces $\mathbb{V}^a_{1,1}$. The first step is to compute $\rho^a_{1,1}$ for $\mathrm{SO}(p)_2$ with the help of Theorem 3.3.2 and Theorem 3.3.4, after which we will find out that, with respect to the standard basis $\{v^a(b)\}$ of $\mathbb{V}^a_{1,1}$, T^a is integral but $\overline{S}^a$ is not. Next we will propose a new basis and prove that, with respect to the new basis, both $\overline{S}^a$ and T^a are integral. Note that, by Theorem 3.3.4 and Lemma 5.0.2 , it is enough to find a basis for $\mathbb{V}^a_{1,1}$ such that S^a and T^a is integral with respect to this basis.

5.1 $Y_{[2k]}$-specialization

For $1 \leq k \leq r$, we consider $\mathbb{V}_{1,1}^{Y_{[2k]}}$. According to the fusion rules, the standard basis for $\mathbb{V}_{1,1}^{Y_{[2k]}}$ is $\{v^{Y_{[2k]}}(Y_k), v^{Y_{[2k]}}(X), v^{Y_{[2k]}}(X')\}$. Since $\mathcal{D} = 2\sqrt{p}$ and by [AFT16],

$$\left(F^{Y_{[2k]}}_{Y_{[2k]}Y_{[2k]}Y_{[2k]}}\right)_{\mathbb{1}\mathbb{1}} = \frac{1}{2}. \tag{5.1.1}$$

Hence, by Equation (4.2.14), for simple objects $b, c \in \{Y_k, X, X'\}$, we have

$$\begin{aligned} S^{Y_{[2k]}}_{cb} &= \tilde{S}^{Y_{[2k]}}_{bc} \cdot \frac{d_c^2 \cdot (F^c_{ccc})_{Y_{[2k]}\mathbb{1}} \cdot \left(F^{Y_{[2k]}}_{Y_{[2k]}Y_{[2k]}Y_{[2k]}}\right)_{\mathbb{1}\mathbb{1}}}{\mathcal{D}\,(F^c_{ccc})_{\mathbb{1}\mathbb{1}}} \\ &= \tilde{S}^{Y_{[2k]}}_{bc} \cdot \frac{d_c^2 \cdot (F^c_{ccc})_{Y_{[2k]}\mathbb{1}}}{4\sqrt{p}\,(F^c_{ccc})_{\mathbb{1}\mathbb{1}}}. \end{aligned} \tag{5.1.2}$$

5.1.1 The auxiliary matrix $\tilde{S}^{Y_{[2k]}}$

By the above equation, we would like to first compute the entries of the auxiliary matrix $\tilde{S}^{Y_{[2k]}}$ using Equation (4.2.13). By symmetry we only need to compute the upper triangular entries. Hence we compute the entries of $\tilde{S}^{Y_{[2k]}}$ in the order of $(Y_k, Y_k), (Y_k, X), (Y_k, X'), (X, X), (X, X'), (X', X')$.

(Y_k, Y_k)**-entry:**

By [AFT16], we have

$$F^{Y_k}_{Y_kY_kY_k} = G^{Y_k}_{Y_kY_kY_k} = \frac{1}{2}\begin{pmatrix} 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}, \tag{5.1.3}$$

and

$$
\begin{aligned}
(R^{\mathbb{1}}_{Y_kY_k})^2 &= (\exp(h_{\mathbb{1}} - 2h_{Y_k}\pi i))^2 \\
&= \exp\left(\frac{-2k(p-k)}{p}\pi i\right) \\
&= \zeta^{k^2},
\end{aligned} \tag{5.1.4}
$$

$$
\begin{aligned}
(R^{Z}_{Y_kY_k})^2 &= (\exp(h_Z - 2h_{Y_k}\pi i))^2 \\
&= (\exp\left(\left(1 - \frac{-k(p-k)}{p}\right)\pi i\right))^2 \\
&= (-\exp\left(\frac{-k(p-k)}{p}\pi i\right))^2 \\
&= \exp\left(\frac{-2k(p-k)}{p}\pi i\right) \\
&= \zeta^{k^2},
\end{aligned} \tag{5.1.5}
$$

$$
\begin{aligned}
(R^{Y_{[2k]}}_{Y_kY_k})^2 &= (\exp\left(h_{Y_{[2k]}} - 2h_{Y_k}\pi i\right))^2 \\
&= \exp\left(2 \cdot \left(\frac{2k(p-2k)}{2p} - \frac{2k(p-k)}{2p}\right)\pi i\right) \\
&= \exp\left(\frac{2(kp - 2k^2 - kp + k^2)}{p}\pi i\right) \\
&= \exp\left(\frac{-2k^2}{p}\pi i\right) \\
&= \zeta^{-k^2}.
\end{aligned} \tag{5.1.6}
$$

Using the above formulas, we determine the (Y_k, Y_k)-entry of $\tilde{S}^{Y_{[2k]}}$:

$$
\begin{aligned}
\tilde{S}^{Y_{[2k]}}_{Y_k,Y_k} &= \sum_j R^j_{Y_kY_k} R^j_{Y_kY_k} (F^{Y_k}_{Y_kY_kY_k})_{\mathbb{1}j} (G^{Y_k}_{Y_kY_kY_k})_{jY_{[2k]}} \\
&= (R^{\mathbb{1}}_{Y_kY_k})^2 (F^{Y_k}_{Y_kY_kY_k})_{\mathbb{1}\mathbb{1}} (G^{Y_k}_{Y_kY_kY_k})_{\mathbb{1}Y_{[2k]}} \\
&\quad + (R^{Z}_{Y_kY_k})^2 (F^{Y_k}_{Y_kY_kY_k})_{\mathbb{1}Z} (G^{Y_k}_{Y_kY_kY_k})_{ZY_{[2k]}} \\
&\quad + (R^{Y_k}_{Y_kY_k})^2 (F^{Y_k}_{Y_kY_kY_k})_{\mathbb{1}Y_k} (G^{Y_k}_{Y_kY_kY_k})_{Y_kY_{[2k]}} \\
&= \zeta^{k^2} \cdot (\frac{1}{2}) \cdot (\frac{1}{\sqrt{2}}) + \zeta^{k^2} \cdot (\frac{-1}{2}) \cdot (\frac{1}{\sqrt{2}}) + \zeta^{-k^2} \cdot (\frac{\sqrt{2}}{2}) \cdot 0 \\
&= 0.
\end{aligned}
\tag{5.1.7}
$$

(Y_k, X)-entry:

We have

$$
F^X_{Y_kY_kX} = G^X_{Y_kY_kX} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5.1.8}
$$

We also have

$$
\begin{aligned}
R^X_{Y_kX} R^X_{XY_k} &= \exp\left(2(h_X - h_X - h_{Y_k})\pi i\right) \\
&= \exp\left(-2h_{Y_k}\pi i\right) \\
&= \zeta^{-rk^2}
\end{aligned}
\tag{5.1.9}
$$

and

$$
\begin{aligned}
R^{X'}_{Y_kX}R^{X'}_{XY_k} &= \exp\left(2(h_{X'}-h_X-h_{Y_k})\pi i\right) \\
&= \exp\left(2(\frac{1}{2}-h_{Y_k})\pi i\right) \\
&= \exp\left(\left(1-2h_{Y_k}\right)\pi i\right) \\
&= (-1)\cdot\exp\left(-2h_{Y_k}\pi i\right) \\
&= -\zeta^{-rk^2}.
\end{aligned}
\tag{5.1.10}
$$

Therefore, we have

$$
\begin{aligned}
\tilde{S}^{Y_{[2k]}}_{Y_k,X} &= R^X_{Y_kX}R^X_{XY_k}(F^X_{Y_kY_kX})_{\mathbb{1}X}(G^X_{Y_kY_kX})_{XY_{[2k]}} \\
&\quad + R^{X'}_{Y_kX}R^{X'}_{XY_k}(F^X_{Y_kY_kX})_{\mathbb{1}X'}(G^X_{Y_kY_kX})_{X'Y_{[2k]}} \\
&= \zeta^{-rk^2}\cdot(\frac{1}{\sqrt{2}})\cdot(\frac{1}{\sqrt{2}})+(-\zeta^{-rk^2})\cdot(\frac{1}{\sqrt{2}})\cdot(-\frac{1}{\sqrt{2}}) \\
&= \zeta^{-rk^2}.
\end{aligned}
\tag{5.1.11}
$$

(Y_k, X')**-entry:**

We have

$$
F^{X'}_{Y_kY_kX'} = G^{X'}_{Y_kY_kX'} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5.1.12}
$$

We also have

$$
\begin{aligned}
R^X_{Y_kX'}R^X_{X'Y_k} &= \exp\left(2(h_X-h_{X'}-h_{Y_k})\pi i\right) \\
&= \exp\left((-1-2h_{Y_k})\pi i\right) \\
&= -\zeta^{-rk^2}
\end{aligned}
\tag{5.1.13}
$$

and

$$\begin{aligned} R^{X'}_{Y_kX'}R^{X'}_{X'Y_k} &= \exp\left(2(h_{X'} - h_{X'} - h_{Y_k})\pi i\right) \\ &= \exp\left(-2h_{Y_k}\pi i\right) \\ &= \zeta^{-rk^2}. \end{aligned} \tag{5.1.14}$$

Therefore, we have

$$\begin{aligned} \tilde{S}^{Y_{[2k]}}_{Y_k,X'} = & R^{X}_{Y_kX'}R^{X}_{X'Y_k}(F^{X'}_{Y_kY_kX'})_{\mathbb{1}X}(G^{X'}_{Y_kY_kX'})_{XY_{[2k]}} \\ & + R^{X'}_{Y_kX'}R^{X'}_{X'Y_k}(F^{X'}_{Y_kY_kX'})_{\mathbb{1}X'}(G^{X'}_{Y_kY_kX'})_{X'Y_{[2k]}} \\ = & -\zeta^{-rk^2}\cdot(\frac{1}{\sqrt{2}})\cdot(\frac{1}{\sqrt{2}}) + (\zeta^{-rk^2})\cdot(\frac{1}{\sqrt{2}})\cdot(-\frac{1}{\sqrt{2}}) \\ = & -\zeta^{-rk^2}. \end{aligned} \tag{5.1.15}$$

(X, X)**-entry:**

The size of F^X_{XXX} (as well as its inverse G^X_{XXX}) is $(r+1)\times(r+1)$, but we only need one column and one row to compute the S-coefficients. Recall that $q = \exp\left(\frac{1}{p}\pi i\right)$. The values of the relevant entries are given by (see [AFT16])

$$(F^X_{XXX})_{\mathbb{1}\mathbb{1}} = (-1)^0\cdot\mathrm{Re}(\frac{\kappa_p\cdot q^0}{\sqrt{p}}) = \frac{\kappa_p}{\sqrt{p}}, \tag{5.1.16}$$

$$(F^X_{XXX})_{\mathbb{1}Y_j} = (-1)^0\cdot\mathrm{Re}(\frac{\sqrt{2}\kappa_p\cdot q^0}{\sqrt{p}}) = \frac{\sqrt{2}\kappa_p}{\sqrt{p}}, \tag{5.1.17}$$

$$\begin{aligned} (F^X_{XXX})_{Y_jY_{[2k]}} &= (-1)^{2jk}\cdot\mathrm{Re}(\frac{2\kappa_p\cdot q^{2jk}}{\sqrt{p}}) = \frac{2\kappa_p}{\sqrt{p}}\cdot\mathrm{Re}(q^{2jk}) \\ &= \frac{\kappa_p}{\sqrt{p}}\cdot(\zeta^{jk} + \zeta^{-jk}),\ \forall 1\le j\le r, \end{aligned} \tag{5.1.18}$$

where $\mathrm{Re}(z)$ denote the real part of a complex number z.

We also need the values of the R-matrices. We have

$$
\begin{aligned}
(R^{Y_j}_{XX})^2 &= \exp\left(2(h_{Y_j} - 2h_X)\pi i\right) \\
&= \exp\left(\frac{-r}{2}\pi i\right) \cdot \exp\left(2h_{Y_j}\pi i\right) \\
&= \exp\left(\frac{-r}{2}\pi i\right) \cdot \zeta^{rj^2}
\end{aligned} \tag{5.1.19}
$$

for all $1 \le j \le r$, and we have

$$
(R^{\mathbb{1}}_{XX})^2 = \exp\left(2(h_{\mathbb{1}} - 2h_X)\pi i\right) = \exp\left(-4h_X\pi i\right) = \exp\left(\frac{-r}{2}\pi i\right). \tag{5.1.20}
$$

According to [AFT16], F^X_{XXX} is unitary, symmetric and real, so F^X_{XXX} is its own inverse. In other words, $G^X_{XXX} = F^X_{XXX}$. Recall that $\kappa_p^2 = 1$. We have

$$
\begin{aligned}
\tilde{S}^{Y_{[2k]}}_{XX} &= \sum_{j=0}^{r} (R^{Y_j}_{XX})^2 (F^X_{XXX})_{\mathbb{1}Y_j} (G^X_{XXX})_{Y_j Y_{[2k]}} \\
&= \exp\left(\frac{-r}{2}\pi i\right) \cdot \frac{\kappa_p}{\sqrt{p}} \cdot \frac{\sqrt{2}\kappa_p}{\sqrt{p}} \\
&\quad + \sum_{j=1}^{r} \exp\left(\frac{-r}{2}\pi i\right) \zeta^{rj^2} \cdot \frac{\sqrt{2}\kappa_p}{\sqrt{p}} \cdot \frac{\kappa_p}{\sqrt{p}} \cdot (\zeta^{jk} + \zeta^{-jk}) \\
&= \frac{\sqrt{2}}{p} \cdot \exp\left(\frac{-r}{2}\pi i\right) \cdot \left(1 + \sum_{j=1}^{r} \zeta^{rj^2+jk} + \sum_{j=1}^{r} \zeta^{r(-j)^2+(-j)k}\right) \\
&= \frac{\sqrt{2}}{p} \cdot \exp\left(\frac{-r}{2}\pi i\right) \cdot \sum_{j=0}^{p-1} \zeta^{rj^2+jk}.
\end{aligned} \tag{5.1.21}
$$

Since $p = 2r + 1$ and $\zeta^p = 1$, we have $\zeta^{jk} = \zeta^{-2rjk}$. Therefore,

$$
\zeta^{rj^2+jk} = \zeta^{rj^2-2rjk} = \zeta^{r(j-k)^2-rk^2} = \zeta^{-rk^2} \cdot \zeta^{r(j-k)^2}. \tag{5.1.22}
$$

By the quadratic Gauss sum formula,

$$\begin{aligned}\sum_{j=0}^{p-1}\zeta^{rj^2+jk} &= \zeta^{-rk^2}\cdot\sum_{j=0}^{p-1}\zeta^{r(j-k)^2} \\ &= \zeta^{-rk^2}\cdot\epsilon_p\cdot\left(\frac{r}{p}\right)_L\cdot\sqrt{p},\end{aligned} \tag{5.1.23}$$

where ϵ_p is given in Equation (4.2.33).

Therefore, we have

$$\begin{aligned}\tilde{S}_{XX}^{Y_{[2k]}} &= \frac{\sqrt{2}}{p}\cdot\exp\left(\frac{-r}{2}\pi i\right)\cdot\zeta^{-rk^2}\cdot\epsilon_p\cdot\left(\frac{r}{p}\right)_L\cdot\sqrt{p} \\ &= \frac{\sqrt{2}}{\sqrt{p}}\cdot\zeta^{-rk^2}\cdot\exp\left(\frac{-r}{2}\pi i\right)\cdot\epsilon_p\cdot\left(\frac{r}{p}\right)_L.\end{aligned} \tag{5.1.24}$$

By Lemma 4.2.2 , we have

$$\tilde{S}_{XX}^{Y_{[2k]}} = \frac{\sqrt{2}}{\sqrt{p}}\cdot\zeta^{-rk^2}. \tag{5.1.25}$$

(X, X')**-entry:**

By Equation (30), (40) and (42) in [AFT16], we have

$$(F_{XXX'}^{X'})_{ef} = (-1)^{\delta_{e0}+\delta_{f0}+1}\cdot(F_{XXX}^{X})_{ef}. \tag{5.1.26}$$

Note also that, by [AFT16], $F_{XXX'}^{X'}$ is unitary, symmetric and real, so we have $F_{XXX'}^{X'} = G_{XXX'}^{X'}$. Therefore, we have

$$(F_{XXX'}^{X'})_{\mathbb{1}Z} = (-1)^{1+1+1}\cdot(F_{XXX}^{X})_{\mathbb{1}\mathbb{1}} = (-1)\cdot\frac{\kappa_p}{\sqrt{p}}, \tag{5.1.27}$$

and

$$(F_{XXX'}^{X'})_{\mathbb{1}Y_j} = (G_{XXX'}^{X'})_{ZY_j} = (-1)^{1+0+1}\cdot(F_{XXX}^{X})_{\mathbb{1}Y_j} = \frac{\sqrt{2}\kappa_p}{\sqrt{p}}, \tag{5.1.28}$$

where the ZY_j indices in the G-matrix is given by the fusion rules. We also have

$$\begin{aligned}(F^{X'}_{XXX'})_{Y_jY_{[2k]}} &= (-1)^{0+0+1} \cdot (F^{X}_{XXX})_{Y_jY_{[2k]}} \\ &= (-1) \cdot \frac{\kappa_p}{\sqrt{p}} \cdot \left(\zeta^{jk} + \zeta^{-jk}\right).\end{aligned} \tag{5.1.29}$$

By Corollary 4.2.1 , we have

$$\begin{aligned}R^{Z}_{XX'} \cdot R^{Z}_{X'X} &= \exp\left(-2\left(h_X + h_{X'} - h_Z\right)\pi i\right) \\ &= \exp\left(\frac{-2(r+r+4)}{8}\pi i\right) \cdot \exp\left(2\pi i\right) \\ &= (-1) \cdot \exp\left(\frac{-r}{2}\pi i\right),\end{aligned} \tag{5.1.30}$$

For any $1 \leq j \leq r$ we also have

$$\begin{aligned}R^{Y_j}_{XX'} \cdot R^{Y_j}_{X'X} &= \exp\left(-2\left(h_X + h_{X'} - h_{Y_j}\right)\pi i\right) \\ &= \exp\left(\frac{-2(r+r+4)}{8}\pi i\right) \cdot \exp\left(2h_{Y_j}\pi i\right) \\ &= (-1) \cdot \exp\left(\frac{-r}{2}\pi i\right) \cdot \zeta^{rj^2}.\end{aligned} \tag{5.1.31}$$

Therefore, we have

$$
\begin{aligned}
\tilde{S}^{Y_{[2k]}}_{XX'} &= R^{Z}_{XX'} \cdot R^{Z}_{X'X} \cdot (F^{X'}_{XXX'})_{\mathbb{1}Z} \cdot (G^{X'}_{XXX'})_{ZY_{[2k]}} \\
&\quad + \sum_{j=1}^{r} R^{Y_j}_{XX'} \cdot R^{Y_j}_{X'X} \cdot (F^{X'}_{XXX'})_{\mathbb{1}Y_j} \cdot (G^{X'}_{XXX'})_{Y_jY_{[2k]}} \\
&= (-1) \cdot \exp\left(\frac{-r}{2}\pi i\right) \cdot (-1) \cdot \frac{\kappa_p}{\sqrt{p}} \cdot \frac{\sqrt{2}\kappa_p}{\sqrt{p}} \\
&\quad + \sum_{j=1}^{r} (-1) \cdot \exp\left(\frac{-r}{2}\pi i\right) \cdot \zeta^{rj^2} \cdot \frac{\sqrt{2}\kappa_p}{\sqrt{p}} \cdot (-1) \cdot \frac{\kappa_p}{\sqrt{p}} \cdot (\zeta^{jk} + \zeta^{-jk}) \\
&= \exp\left(\frac{-r}{2}\pi i\right) \cdot \frac{\kappa_p}{\sqrt{p}} \cdot \frac{\sqrt{2}\kappa_p}{\sqrt{p}} \\
&\quad + \sum_{j=1}^{r} \exp\left(\frac{-r}{2}\pi i\right) \cdot \zeta^{rj^2} \cdot \frac{\sqrt{2}\kappa_p}{\sqrt{p}} \cdot \frac{\kappa_p}{\sqrt{p}} \cdot (\zeta^{jk} + \zeta^{-jk}) \\
&= \frac{\sqrt{2}}{\sqrt{p}} \cdot \zeta^{-rk^2},
\end{aligned}
\tag{5.1.32}
$$

where the last equality is based on the same computation in Equations (5.1.21) to (5.1.25).

(X', X')-entry:

By [AFT16], $F^{X'}_{X'X'X'} = F^{X}_{XXX}$ and $G^{X'}_{X'X'X'} = G^{X}_{XXX}$. Furthermore, we have, for any $1 \le j \le r$,

$$
\begin{aligned}
(R^{Y_j}_{X',X'})^2 &= \exp\left(-2\left(2h_{X'} - h_{Y_j}\right)\pi i\right) \\
&= \exp\left(\frac{-2 \cdot 2(r+4)}{8}\pi i\right) \cdot \exp\left(2h_{Y_j}\pi i\right) \\
&= \exp\left(\frac{-r}{2}\pi i\right) \cdot \zeta^{rj^2} \\
&= (R^{Y_j}_{XX})^2.
\end{aligned}
\tag{5.1.33}
$$

Therefore, we have

$$
\begin{aligned}
\tilde{S}^{Y_{[2k]}}_{X'X'} &= \sum_{j=0}^{r} (R^{Y_j}_{X'X'})^2 \cdot (F^{X'}_{X'X'X'})_{\mathbb{1}Y_j} \cdot (G^{X'}_{X'X'X'})_{Y_jY_{[2k]}} \\
&= \sum_{j=0}^{r} (R^{Y_j}_{XX})^2 (F^{X}_{XXX})_{\mathbb{1}Y_j} (G^{X}_{XXX})_{Y_jY_{[2k]}} \\
&= \frac{\sqrt{2}}{\sqrt{p}} \cdot \zeta^{-rk^2}.
\end{aligned} \tag{5.1.34}
$$

Again, the last equality is based on the same computation in Equations (5.1.21) to (5.1.25).

We summarize the above computations in the following proposition.

Proposition 5.1.1. *The matrix presentation of the auxiliary operator $\tilde{S}^{Y_{[2k]}}$ with respect to the standard basis for $\mathbb{V}^{Y_{[2k]}}_{1,1}$ is*

$$
\tilde{S}^{Y_{[2k]}} = \zeta^{-rk^2} \cdot \begin{pmatrix} 0 & 1 & -1 \\ 1 & \frac{\sqrt{2}}{\sqrt{p}} & \frac{\sqrt{2}}{\sqrt{p}} \\ -1 & \frac{\sqrt{2}}{\sqrt{p}} & \frac{\sqrt{2}}{\sqrt{p}} \end{pmatrix}. \tag{5.1.35}
$$

5.1.2 The action of $S^{Y_{[2k]}}$

We now compute the $S^{Y_{[2k]}}$-action. Note that by Equation (5.1.3),

$$
\left(F^{Y_k}_{Y_kY_kY_k}\right)_{Y_{[2k]}\mathbb{1}} = \frac{\sqrt{2}}{2}, \tag{5.1.36}
$$

and by Equation (5.1.17), the fact that $F^{X}_{XXX} = F^{X'}_{X'X'X'}$, and that theses F-matrices are symmetric [AFT16], we have

$$\left(F^X_{XXX}\right)_{Y_{[2k]}\mathbb{1}} = \left(F^{X'}_{X'X'X'}\right)_{Y_{[2k]}\mathbb{1}} = \left(F^X_{XXX}\right)_{\mathbb{1}Y_{[2k]}} = \frac{\sqrt{2}\kappa_p}{\sqrt{p}}. \tag{5.1.37}$$

With the above formulas and Proposition 5.1.1 we can use Equation (5.1.2) to compute

$$S^{Y_{[2k]}}_{Y_kY_k} = \tilde{S}^{Y_{[2k]}}_{Y_kY_k} \cdot \frac{d^2_{Y_k} \cdot \left(F^{Y_k}_{Y_kY_kY_k}\right)_{Y_{[2k]}\mathbb{1}}}{4\sqrt{p}\left(F^{Y_k}_{Y_kY_kY_k}\right)_{\mathbb{1}\mathbb{1}}} = 0, \tag{5.1.38}$$

$$\begin{aligned} S^{Y_{[2k]}}_{XY_k} &= \tilde{S}^{Y_{[2k]}}_{Y_kX} \cdot \frac{d^2_X \cdot \left(F^X_{XXX}\right)_{Y_{[2k]}\mathbb{1}}}{4\sqrt{p}\,(F^X_{XXX})_{\mathbb{1}\mathbb{1}}} = \zeta^{-rk^2} \cdot \frac{\left(\sqrt{p}\right)^2 \cdot \left(\dfrac{\sqrt{2}\kappa_p}{\sqrt{p}}\right)}{4\sqrt{p} \cdot \left(\dfrac{\kappa_p}{\sqrt{p}}\right)} \\ &= \frac{\sqrt{2p}}{4} \cdot \zeta^{-rk^2}, \end{aligned} \tag{5.1.39}$$

and

$$\begin{aligned} S^{Y_{[2k]}}_{X'Y_k} &= \tilde{S}^{Y_{[2k]}}_{Y_kX'} \cdot \frac{d^2_{X'} \cdot \left(F^{X'}_{X'X'X'}\right)_{Y_{[2k]}\mathbb{1}}}{4\sqrt{p}\left(F^{X'}_{X'X'X'}\right)_{\mathbb{1}\mathbb{1}}} = (-\zeta^{-rk^2}) \cdot \frac{\left(\sqrt{p}\right)^2 \cdot \left(\dfrac{\sqrt{2}\kappa_p}{\sqrt{p}}\right)}{4\sqrt{p} \cdot \left(\dfrac{\kappa_p}{\sqrt{p}}\right)} \\ &= -\frac{\sqrt{2p}}{4} \cdot \zeta^{-rk^2}. \end{aligned} \tag{5.1.40}$$

For the other entries, we first of all have

$$\begin{aligned} S^{Y_{[2k]}}_{Y_kX} &= \tilde{S}^{Y_{[2k]}}_{XY_k} \cdot \frac{d^2_{Y_k} \cdot \left(F^{Y_k}_{Y_kY_kY_k}\right)_{Y_{[2k]}\mathbb{1}}}{4\sqrt{p}\left(F^{Y_k}_{Y_kY_kY_k}\right)_{\mathbb{1}\mathbb{1}}} = \zeta^{-rk^2} \cdot \frac{2^2 \cdot \left(\dfrac{\sqrt{2}}{2}\right)}{4\sqrt{p} \cdot \left(\dfrac{1}{2}\right)} \\ &= \frac{\sqrt{2}}{\sqrt{p}} \cdot \zeta^{-rk^2}. \end{aligned} \tag{5.1.41}$$

and

$$S^{Y_{[2k]}}_{Y_k X'} = \tilde{S}^{Y_{[2k]}}_{X' Y_k} \cdot \frac{d^2_{Y_k} \cdot \left(F^{Y_k}_{Y_k Y_k Y_k}\right)_{Y_{[2k]}\mathbb{1}}}{4\sqrt{p}\left(F^{Y_k}_{Y_k Y_k Y_k}\right)_{\mathbb{1}\mathbb{1}}} = \left(-\zeta^{-rk^2}\right) \cdot \frac{2^2 \cdot \left(\dfrac{\sqrt{2}}{2}\right)}{4\sqrt{p} \cdot \left(\dfrac{1}{2}\right)}$$
$$= -\frac{\sqrt{2}}{\sqrt{p}} \cdot \zeta^{-rk^2}. \tag{5.1.42}$$

Finally, by Proposition (5.1.1), we have

$$\tilde{S}^{Y_{[2k]}}_{XX} = \tilde{S}^{Y_{[2k]}}_{XX'} = \tilde{S}^{Y_{[2k]}}_{X'X} = \tilde{S}^{Y_{[2k]}}_{X'X'}. \tag{5.1.43}$$

Equations (5.1.16) and (5.1.37), together with the fact that $d_X = d_{X'} = \sqrt{p}$, imply that

$$S^{Y_{[2k]}}_{XX} = S^{Y_{[2k]}}_{XX'} = S^{Y_{[2k]}}_{X'X} = S^{Y_{[2k]}}_{X'X'} = \tilde{S}^{Y_{[2k]}}_{XX} \cdot \frac{d^2_X \cdot \left(F^X_{XXX}\right)_{Y_{[2k]}\mathbb{1}}}{4\sqrt{p}\left(F^X_{XXX}\right)_{\mathbb{1}\mathbb{1}}}$$
$$= \zeta^{-rk^2} \cdot \frac{\sqrt{2}}{\sqrt{p}} \cdot \frac{\left(\sqrt{p}\right)^2 \cdot \left(\dfrac{\sqrt{2}\kappa_p}{\sqrt{p}}\right)}{4\sqrt{p} \cdot \left(\dfrac{\kappa_p}{\sqrt{p}}\right)} \tag{5.1.44}$$
$$= \frac{1}{2} \cdot \zeta^{-rk^2}.$$

We summarize the above computations in the following proposition.

Proposition 5.1.2. *The matrix presentation of* $S^{Y_{[2k]}}$ *with respect to the standard basis for* $\mathbb{V}^{Y_{[2k]}}_{1,1}$ *is*

$$S^{Y[2k]} = \zeta^{-rk^2} \cdot \begin{pmatrix} 0 & \dfrac{\sqrt{2}}{\sqrt{p}} & -\dfrac{\sqrt{2}}{\sqrt{p}} \\ \dfrac{\sqrt{2p}}{4} & \dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{\sqrt{2p}}{4} & \dfrac{1}{2} & \dfrac{1}{2} \end{pmatrix}. \tag{5.1.45}$$

5.1.3 The action of $T^{Y[2k]}$

Recall that $\psi = \theta_X = \exp\left(\frac{r}{4}\pi i\right)$ (see Equation (4.2.17)). By Theorem 3.3.2 and Equations (4.2.18) and (4.2.20), we have the following proposition.

Proposition 5.1.3. *The matrix presentation of the meridian twist operator $T^{Y[2k]}$ with respect to the standard basis for $\mathbb{V}_{1,1}^{Y[2k]}$ is*

$$T^{Y[2k]} = \begin{pmatrix} \zeta^{rk^2} & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & -\psi \end{pmatrix}. \tag{5.1.46}$$

5.1.4 Integral basis for $\mathbb{V}_{1,1}^{Y[2k]}$

Combing the results of this section, we are now ready to give an integral basis of $\mathbb{V}_{1,1}^{Y[2k]}$. Let

$$\begin{aligned} w_1^{Y[2k]} &= \frac{2\sqrt{2}}{\sqrt{p}} \cdot v^{Y[2k]}(Y_k), \\ w_2^{Y[2k]} &= \psi\left(v^{Y[2k]}(X) + v^{Y[2k]}(X')\right), \\ w_3^{Y[2k]} &= v^{Y[2k]}(X) - v^{Y[2k]}(X'), \end{aligned} \tag{5.1.47}$$

we have

Theorem 5.1.1. $\{w_1^{Y_{[2k]}}, w_2^{Y_{[2k]}}, w_3^{Y_{[2k]}}\}$ *is an integral basis for* $\mathbb{V}_{1,1}^{Y_{[2k]}}$.

Proof. Clearly, the set in the theorem constitutes a basis for $\mathbb{V}_{1,1}^{Y_{[2k]}}$. We prove the theorem by direct computation of the mapping class group action on these basis elements. For the $S^{Y_{[2k]}}$ action, we have

$$\begin{aligned}
&S^{Y_{[2k]}}\left(w_1^{Y_{[2k]}}\right)\\
&=\frac{2\sqrt{2}}{\sqrt{p}}\cdot\zeta^{-rk^2}\cdot\left(\frac{\sqrt{2p}}{4}\cdot v^{Y_{[2k]}}(X)-\frac{\sqrt{2p}}{4}v^{Y_{[2k]}}(X')\right)\\
&=\zeta^{-rk^2}\cdot\left(v^{Y_{[2k]}}(X)-v^{Y_{[2k]}}(X')\right)\\
&=\zeta^{-rk^2}\cdot w_3^{Y_{[2k]}},
\end{aligned}\tag{5.1.48}$$

$$\begin{aligned}
&S^{Y_{[2k]}}\left(w_2^{Y_{[2k]}}\right)\\
&=\psi\cdot\zeta^{-rk^2}\cdot\left(\frac{\sqrt{2}}{\sqrt{p}}-\frac{\sqrt{2}}{\sqrt{p}}\right)\cdot v^{Y_{[2k]}}(Y_k)\\
&\quad+\psi\cdot\zeta^{-rk^2}\cdot\left(\left(\frac{1}{2}+\frac{1}{2}\right)\cdot v^{Y_{[2k]}}(X)+\left(\frac{1}{2}+\frac{1}{2}\right)\cdot v^{Y_{[2k]}}(X')\right)\\
&=\zeta^{-rk^2}\cdot\psi\cdot\left(v^{Y_{[2k]}}(X)+v^{Y_{[2k]}}(X')\right)\\
&=\zeta^{-rk^2}\cdot w_2^{Y_{[2k]}},
\end{aligned}\tag{5.1.49}$$

and

$$
\begin{aligned}
&S^{Y_{[2k]}}\left(w_3^{Y_{[2k]}}\right)\\
=&\zeta^{-rk^2}\cdot\left(\frac{\sqrt{2}}{\sqrt{p}}+\frac{\sqrt{2}}{\sqrt{p}}\right)\cdot v^{Y_{[2k]}}(Y_k)\\
&+\zeta^{-rk^2}\cdot\left(\left(\frac{1}{2}-\frac{1}{2}\right)\cdot v^{Y_{[2k]}}(X)+\left(\frac{1}{2}-\frac{1}{2}\right)\cdot v^{Y_{[2k]}}(X')\right)\\
=&\zeta^{-rk^2}\cdot\frac{2\sqrt{2}}{\sqrt{p}}\cdot v^{Y_{[2k]}}(Y_k)\\
=&\zeta^{-rk^2}\cdot w_1^{Y_{[2k]}}.
\end{aligned}
\tag{5.1.50}
$$

In other words, with respect to the basis $\{w_1^{Y_{[2k]}}, w_2^{Y_{[2k]}}, w_3^{Y_{[2k]}}\}$, the matrix presentation of $S^{Y_{[2k]}}$, denoted by $\mathcal{S}^{Y_{[2k]}}$, is

$$
\mathcal{S}^{Y_{[2k]}}=\zeta^{-rk^2}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},
\tag{5.1.51}
$$

which is integral.

Similarly, we have

$$
\begin{aligned}
T^{Y_{[2k]}}\left(w_1^{Y_{[2k]}}\right)&=\zeta^{rk^2}\cdot\frac{2\sqrt{2}}{\sqrt{p}}v^{Y_{[2k]}}(Y_k)\\
&=\zeta^{rk^2}\cdot w_1^{Y_{[2k]}},
\end{aligned}
\tag{5.1.52}
$$

$$
\begin{aligned}
T^{Y_{[2k]}}\left(w_2^{Y_{[2k]}}\right)&=\psi\cdot\left(\psi\cdot v^{Y_{[2k]}}(X)-\psi\cdot v^{Y_{[2k]}}(X')\right)\\
&=\psi^2(v^{Y_{[2k]}}(X)-v^{Y_{[2k]}}(X'))\\
&=\psi^2\cdot w_3^{Y_{[2k]}}=i^r\cdot w_3^{Y_{[2k]}},
\end{aligned}
\tag{5.1.53}
$$

and

$$\begin{aligned} T^{Y_{[2k]}}\left(w_3^{Y_{[2k]}}\right) &= \psi \cdot v^{Y_{[2k]}}(X) - (-\psi) \cdot v^{Y_{[2k]}}(X') \\ &= \psi \left(v^{Y_{[2k]}}(X) + v^{Y_{[2k]}}(X')\right) \\ &= w_2^{Y_{[2k]}}. \end{aligned} \tag{5.1.54}$$

Hence the $T^{Y_{[2k]}}$ action with respect to the basis $\{w_1^{Y_{[2k]}}, w_2^{Y_{[2k]}}, w_3^{Y_{[2k]}}\}$, denoted by $\mathcal{T}^{Y_{[2k]}}$, is

$$\mathcal{T}^{Y_{[2k]}} = \begin{pmatrix} \zeta^{rk^2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i^r & 0 \end{pmatrix}. \tag{5.1.55}$$

Therefore, all the entries of $\mathcal{T}^{Y_{[2k]}}$ are in $\mathbb{Z}[\zeta, i]$. Combining the integrality of $\mathcal{S}^{Y_{[2k]}}$ and $\mathcal{T}^{Y_{[2k]}}$, we are done. $\square$

In particular, $\Lambda_{Y_{[2k]}} = \operatorname{span}_{\mathcal{O}}(\{w_1^{Y_{[2k]}}, w_2^{Y_{[2k]}}, w_3^{Y_{[2k]}}\})$ is a $\widetilde{\Gamma_{1,1}}$-invariant full-rank free $\mathcal{O}$-lattice in $\mathbb{V}_{1,1}^{Y_{[2k]}}$.

5.2 $\mathbb{1}$-specialization

Since $\theta_{\mathbb{1}} = 1$, as noted in Section 3.3.4 , the Dehn twist along the boundary curve acts trivially on $\mathbb{V}_{1,1}^{\mathbb{1}}$, hence $\rho_{1,1}^{\mathbb{1}}$ factors through a representation of $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$ on $\mathbb{V}_{1,1}^{\mathbb{1}}$. Note that $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$ is isomorphic to the central extension of the mapping class group of the closed torus $\Sigma_{1,0}$ by framing.

As before, by the fusion rules, $\mathbb{V}_{1,1}^{\mathbb{1}}$ has the standard basis

$$\{v^{\mathbb{1}}(\mathbb{1}), v^{\mathbb{1}}(Z), v^{\mathbb{1}}(Y_1), ..., v^{\mathbb{1}}(Y_r), v^{\mathbb{1}}(X), v^{\mathbb{1}}(X')\}. \tag{5.2.1}$$

Let

$$\mathcal{B}_0 := \{d_j \cdot v^{\mathbb{1}}(j) : j \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)\} \tag{5.2.2}$$

be another basis of $\mathbb{V}^{\mathbb{1}}_{1,1}$. By Proposition 4.2.3 and Proposition 4.2.4 , the matrix presentation of $S^{\mathbb{1}}$ and $T^{\mathbb{1}}$ with respect to $\mathcal{B}_0$ are $\mathfrak{s}$ and $\mathfrak{t}$ respectively, where $\mathfrak{s}$ and $\mathfrak{t}$ are given in Equations (4.2.56) and (4.2.65). Instead of using the standard basis, in this section, we will use the basis $\mathcal{B}_0$ to start our search for an integral basis for $\mathbb{V}^{\mathbb{1}}_{1,1}$. We will show that there is an integral basis for $\mathbb{V}^{\mathbb{1}}_{1,1}$ with respect to which $\mathfrak{s}$ and $\mathfrak{t}$ are integral.

Since the basis vectors in $\mathcal{B}_0$ are in one-to-one correspondence with the simple objects, we will label them by simple objects by an abuse of notation. In other words,

$$\mathcal{B}_0 = \{\mathbb{1}, Z, Y_1, ..., Y_r, X, X'\}. \tag{5.2.3}$$

5.2.1 Integral basis for $\mathbb{V}^{\mathbb{1}}_{1,1}$

The statement of the existence of an integral basis for $\mathbb{V}^{\mathbb{1}}_{1,1}$ is similar to that in last section:

Theorem 5.2.1. *There exists a $\widetilde{\Gamma_{1,1}}$-invariant,full-rank, free $\mathcal{O}$-lattice in $\mathbb{V}^{\mathbb{1}}_{1,1}$.*

To prove the theorem we will give an explicit change of basis matrix $W \in \mathrm{GL}(n_{\mathbb{1}}, \mathbb{C})$ so that $W^{-1}\mathfrak{s}W$ and $W^{-1}\mathfrak{t}W$ are integral. Recall that $n_{\mathbb{1}} = r + 4$. To simplify notations, we write

$$H := \mathbb{V}^{\mathbb{1}}_{1,1}. \tag{5.2.4}$$

We will find W in several steps. First, we decompose H into two $\widetilde{\Gamma_{1,1}}$-invariant subspaces in Lemma 5.2.1 , and reduce the problem to Proposition 5.2.1 . Next, we examine the properties of the column vectors of the representation after the change of basis proposed in Proposition 5.2.1 . Finally, we prove integrality of the first column in Proposition 5.2.3 , and the integrality of the rest of the columns is proved in Proposition 5.2.4 .

Let

$$U = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & & & 1 \\ \hline 0 & \psi & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\psi & 1 & 0 & 0 & \cdots & 0 \end{array}\right) \tag{5.2.5}$$

be an $(r+4)\times(r+4)$-matrix (in particular, the right half of the matrix has r columns). Let $H_1 = \text{span}_{\mathbb{C}}\{\mathbb{1} - Z, \psi(X - X'), X + X'\}$ and $H_2 = \text{span}_{\mathbb{C}}\{\mathbb{1} + Z, Y_1, ..., Y_r\}$ be two subspaces of H.

Lemma 5.2.1. *H decomposes into a direct sum of two $\widetilde{\Gamma_{1,1}}$-invariant subspaces $H \cong H_1 \oplus H_2$. Moreover, $U^{-1}\mathfrak{s}U|_{H_1}$ and $U^{-1}\mathfrak{t}U|_{H_1}$ are integral.*

Proof. The basis of H corresponding to U is $\mathcal{B}_U = \{\mathbb{1} - Z, \psi(X - X'), X + X', \mathbb{1} + Z, Y_1, ..., Y_r\}$. By Equation (4.2.56), we have

$$
\begin{aligned}
\mathfrak{s}(\mathbb{1}) &= \frac{1}{2\sqrt{p}}\mathbb{1} + \frac{1}{2\sqrt{p}}Z + \frac{1}{\sqrt{p}}\sum_{k=1}^{r} Y_k + \frac{1}{2}X + \frac{1}{2}X', \\
\mathfrak{s}(Z) &= \frac{1}{2\sqrt{p}}\mathbb{1} + \frac{1}{2\sqrt{p}}Z + \frac{1}{\sqrt{p}}\sum_{k=1}^{r} Y_k - \frac{1}{2}X - \frac{1}{2}X', \\
\mathfrak{s}(Y_j) &= \frac{1}{\sqrt{p}}\mathbb{1} + \frac{1}{\sqrt{p}}Z + \sum_{k=1}^{r} A_{kj}Y_k, \ \forall j = 1, ..., r, \\
\mathfrak{s}(X) &= \frac{1}{2}\mathbb{1} - \frac{1}{2}Z + \frac{1}{2}X - \frac{1}{2}X', \\
\mathfrak{s}(X') &= \frac{1}{2}\mathbb{1} - \frac{1}{2}Z - \frac{1}{2}X + \frac{1}{2}X',
\end{aligned} \tag{5.2.6}
$$

where A_{kj} is as Equation (4.2.58). So the action of $\mathfrak{s}$ on $\mathcal{B}_U$ basis vectors is given by

$$
\begin{aligned}
\mathfrak{s}(\mathbb{1} - Z) &= X + X', \\
\mathfrak{s}(X + X') &= \mathbb{1} - Z, \\
\mathfrak{s}(\psi(X - X')) &= \psi(X - X'), \\
\mathfrak{s}(\mathbb{1} + Z) &= \frac{1}{\sqrt{p}}(\mathbb{1} + Z) + \frac{2}{\sqrt{p}}\sum_{k=1}^{r} Y_k, \\
\mathfrak{s}(Y_j) &= \frac{1}{\sqrt{p}}(\mathbb{1} + Z) + \sum_{k=1}^{r} A_{kj}Y_k, \ \forall j = 1, ..., r.
\end{aligned} \tag{5.2.7}
$$

Therefore, $\mathfrak{s}(H_j) = H_j$ for $j = 1, 2$. Moreover, with respect to $\mathcal{B}_U$, $\mathfrak{s}$ has matrix presentation

$$U^{-1}\mathfrak{s}U = \left(\begin{array}{ccc|cc} 0 & 0 & 1 & & \\ 0 & 1 & 0 & & \mathbf{0} \\ 1 & 0 & 0 & & \\ \hline & & & \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot \boldsymbol{a}^t \\ & \mathbf{0} & & & \\ & & & \frac{2}{\sqrt{p}} \cdot \boldsymbol{a} & A \end{array}\right), \tag{5.2.8}$$

where $\boldsymbol{a}$ is as in Equation (4.2.57).

Similarly, we have

$$\begin{aligned} &\mathfrak{t}(\mathbb{1} - Z) = \mathbb{1} - Z, \\ &\mathfrak{t}(X + X') = \psi(X - X'), \\ &\mathfrak{t}\left(\psi(X - X')\right) = \psi^2(X + X') = i^r(X + X'), \\ &\mathfrak{t}(\mathbb{1} + Z) = \mathbb{1} + Z, \\ &\mathfrak{t}(Y_j) = \theta_j Y_j, \ \forall j = 1, ..., r, \end{aligned} \tag{5.2.9}$$

where ψ is as in Equation (4.2.17), and θ_j is as in Equation (4.2.20) for any $j = 1, ..., r$.

By the above computations, $\mathfrak{t}(H_j) = H_j$ for $j = 1, 2$. In addition, with respect to $\mathcal{B}_U$, $\mathfrak{t}$ acts as

$$U^{-1}\mathfrak{t}U = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & & \mathbf{0} & & \\ 0 & i^r & 0 & & & & \\ \hline & & & 1 & & & \\ & \mathbf{0} & & & \theta_1 & & \\ & & & & & \ddots & \\ & & & & & & \theta_r \end{array}\right). \tag{5.2.10}$$

Therefore, we have a decomposition of H into $\widetilde{\Gamma_{1,1}}$-invariant subspaces

$$H \cong H_1 \oplus H_2. \tag{5.2.11}$$

Moreover, we have

$$U^{-1}\mathfrak{s}U|_{H_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sqsubset \mathcal{O} \tag{5.2.12}$$

and

$$U^{-1}\mathfrak{t}U|_{H_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i^r & 0 \end{pmatrix} \sqsubset \mathcal{O}. \tag{5.2.13}$$

□

Given Lemma 5.2.1 , it remains to find an integral basis for the $(r+1)$-dimensional subspace H_2. We consider

$$s' := U^{-1}\mathfrak{s}U|_{H_2} = \begin{pmatrix} \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot \boldsymbol{a}^t \\ \frac{2}{\sqrt{p}} \cdot \boldsymbol{a} & A \end{pmatrix} \tag{5.2.14}$$

and

$$t' := U^{-1}\mathfrak{t}U|_{H_2} = \begin{pmatrix} 1 & & & \\ & \theta_1 & & \\ & & \ddots & \\ & & & \theta_r \end{pmatrix}. \tag{5.2.15}$$

Instead of s' and t', we would prefer to work with $\boldsymbol{s}$ and $\boldsymbol{t}$ defined as follows. Let

$$D = \begin{pmatrix} \frac{1}{2} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \tag{5.2.16}$$

we define

$$\boldsymbol{s} := D^{-1}s'D = \begin{pmatrix} \frac{1}{\sqrt{p}} & \frac{2}{\sqrt{p}} \cdot \boldsymbol{a}^t \\ \frac{1}{\sqrt{p}} \cdot \boldsymbol{a} & A \end{pmatrix} \tag{5.2.17}$$

and

$$\boldsymbol{t} := D^{-1}t'D = t' = \begin{pmatrix} 1 & & & \\ & \theta_1 & & \\ & & \ddots & \\ & & & \theta_r \end{pmatrix}. \tag{5.2.18}$$

Let V be the following Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^r \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^r \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \theta_r & \theta_r^2 & \cdots & \theta_r^r \end{pmatrix}. \tag{5.2.19}$$

Note that when for any $1 \leq j \leq r$, $\theta_j = \zeta^{rj^2} \neq 1$. Note also that for any $1 \leq j < k \leq r$, $\theta_j \neq \theta_k$, because otherwise, we would have $(j-k)(j+k) \equiv 0 \mod p$. However, by assumption, $1 \leq j - k \leq r - 1$, and $1 < j + k < 2r$, so $(j-k)(j+k) \not\equiv 0 \mod p$ (recall that $p = 2r + 1$ is a prime), contradiction. In particular, the above argument shows that V is invertible.

Theorem 5.2.1 now follows from the following proposition.

Proposition 5.2.1. *The matrices $V^{-1}\boldsymbol{s}V$ and $V^{-1}\boldsymbol{t}V$ are both integral.*

Proof of Theorem 5.2.1 from Proposition 5.2.1 . Let

$$W = U \cdot \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & DV \end{array}\right). \tag{5.2.20}$$

We have

$$
\begin{aligned}
&W^{-1}\mathfrak{s}W \\
&= \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & V^{-1}D^{-1} \end{array}\right) \cdot (U^{-1}\mathfrak{s}U) \cdot \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & DV \end{array}\right) \\
&= \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & V^{-1}D^{-1} \end{array}\right) \cdot \left(\begin{array}{ccc|c} 0 & 0 & 1 & \\ 0 & 1 & 0 & \mathbf{0} \\ 1 & 0 & 0 & \\ \hline & \mathbf{0} & & s' \end{array}\right) \cdot \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & DV \end{array}\right) \qquad (5.2.21) \\
&= \left(\begin{array}{ccc|c} 0 & 0 & 1 & \\ 0 & 1 & 0 & \mathbf{0} \\ 1 & 0 & 0 & \\ \hline & \mathbf{0} & & V^{-1}D^{-1}s'DV \end{array}\right) \\
&= \left(\begin{array}{ccc|c} 0 & 0 & 1 & \\ 0 & 1 & 0 & \mathbf{0} \\ 1 & 0 & 0 & \\ \hline & \mathbf{0} & & V^{-1}\boldsymbol{s}V \end{array}\right) \sqsubset \mathcal{O}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&W^{-1}\mathfrak{t}W \\
&= \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & V^{-1}D^{-1} \end{array}\right) \cdot (U^{-1}\mathfrak{t}U) \cdot \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & DV \end{array}\right) \\
&= \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & V^{-1}D^{-1} \end{array}\right) \cdot \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & i^r & 0 & \\ \hline & \mathbf{0} & & t' \end{array}\right) \cdot \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \\ \hline & \mathbf{0} & & DV \end{array}\right) \qquad (5.2.22) \\
&= \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & i^r & 0 & \\ \hline & \mathbf{0} & & V^{-1}D^{-1}t'DV \end{array}\right) \\
&= \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & i^r & 0 & \\ \hline & \mathbf{0} & & V^{-1}\boldsymbol{t}V \end{array}\right) \sqsubset \mathcal{O}.
\end{aligned}
$$

□

To prove the proposition we need to study the properties of $\boldsymbol{s}V$. In the following discussion, we index the matrix entries from 0. We write

$$\theta_0 := \zeta^{r\cdot 0^2} = 1. \qquad (5.2.23)$$

For all $j \in \mathbb{Z}$ for which $g.c.d(j,p) = 1$, j is invertible in $\mathbb{Z}/p\mathbb{Z}$. In the following, we understand the exponent $\frac{1}{j}$ of ζ in the expression $\zeta^{\frac{1}{j}}$ as the reciprocal of j in $\mathbb{Z}/p\mathbb{Z}$.

Proposition 5.2.2. *The (j,k)-th matrix coefficient of $\boldsymbol{s}V$ is given by*

$$(\boldsymbol{s}V)_{jk} = \begin{cases} \sqrt{p}, & \text{if } j = k = 0, \\ 0, & \text{if } 1 \leq j \leq r, \text{ and } k = 0, \\ \left(\frac{rk}{p}\right)_L \cdot \epsilon_p \cdot \theta_j^{-\frac{1}{k}}, & \text{if } k \geq 1, \end{cases} \tag{5.2.24}$$

where ϵ_p and $\left(\frac{*}{*}\right)_L$ are as in Equation (4.2.33).

Proof. **Case 1.** When $j = k = 0$, a direct computation shows that

$$(\boldsymbol{s}V)_{00} = \frac{1}{\sqrt{p}} + \frac{2}{\sqrt{p}} \cdot r = \frac{1}{\sqrt{p}} \cdot p = \sqrt{p}. \tag{5.2.25}$$

Case 2. When $1 \leq j \leq r$, and $k = 0$, we have

$$\begin{aligned} (\boldsymbol{s}V)_{j0} =& \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} A_{jl} \cdot 1 \\ =& \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} \frac{1}{\sqrt{p}}(\zeta^{jl} + \zeta^{-jl}) \\ =& \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{jl} \\ =& 0. \end{aligned} \tag{5.2.26}$$

The third equality results from the fact that $-l \equiv p - l \mod p$. The last equality uses that for any $1 \leq j \leq r$, ζ^j is an p-th root of unity, so it is a zero of the minimal polynomial $\Phi_p(x) = 1 + x + \cdots + x^{2r}$.

Case 3. When $1 \leq k \leq r$, we have

$$
\begin{aligned}
(\boldsymbol{s}V)_{jk} &= \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} \frac{1}{\sqrt{p}}(\zeta^{jl} + \zeta^{-jl}) \cdot \theta_l^k \\
&= \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{p}} \sum_{l=1}^{r}(\zeta^{jl} + \zeta^{-jl}) \cdot \zeta^{rkl^2} \\
&= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{jl+rkl^2} \\
&= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{rk(l^2+\frac{j}{rk}l)}.
\end{aligned}
\tag{5.2.27}
$$

Note that, by assumption, $1 \leq k \leq r$. Hence, $\frac{j}{rk}$ is well-defined in the finite field $\mathbb{Z}/p\mathbb{Z}$. Let $\gamma = \frac{j}{2rk} \in \mathbb{Z}/p\mathbb{Z}$, we have

$$
\begin{aligned}
(\boldsymbol{s}V)_{jk} &= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{rk(l^2+2\gamma l)} \\
&= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{rk(l+\gamma)^2 - rk\gamma^2} \\
&= \frac{1}{\sqrt{p}} \zeta^{-rk\gamma^2} \sum_{l=0}^{2r} \zeta^{rk(l+\gamma)^2}.
\end{aligned}
\tag{5.2.28}
$$

Hence, by the quadratic Gauss sum formula (4.2.39), we have

$$
\begin{aligned}
(\boldsymbol{s}V)_{jk} &= \frac{1}{\sqrt{p}} \cdot \zeta^{-rk\gamma^2} \cdot \left(\frac{rk}{p}\right)_L \cdot \epsilon_p \cdot \sqrt{p} \\
&= \left(\frac{rk}{p}\right)_L \cdot \epsilon_p \cdot \zeta^{-rk\gamma^2}.
\end{aligned}
\tag{5.2.29}
$$

Note that

$$
4r^2 - 1 = (2r+1)(2r-1) = p(2r-1) \equiv 0 \mod p, \tag{5.2.30}
$$

we have

$$
k\gamma^2 = \frac{j^2}{4r^2k} \equiv \frac{j^2}{k} \mod p, \tag{5.2.31}
$$

and, consequently,

$$(\boldsymbol{s}V)_{jk} = \left(\frac{rk}{p}\right)_L \cdot \epsilon_p \cdot \zeta^{(rj^2)\cdot(-\frac{1}{k})} = \left(\frac{rk}{p}\right)_L \cdot \epsilon_p \cdot \theta_j^{-\frac{1}{k}}, \tag{5.2.32}$$

as desired. □

To proceed further, let us recall some basic facts in number theory.

Lemma 5.2.2. *Let $\mathcal{O}^\times$ be the group of units in $\mathcal{O}$. Then*

$$p = 2r+1 = (-1)^r \cdot \zeta^{-\frac{r(r+1)}{2}} \cdot \prod_{k=1}^{r} (1-\zeta^k)^2. \tag{5.2.33}$$

Proof. Recall that

$$\Phi_p(x) = 1 + x + \cdots + x^{2r} = \prod_{l=1}^{2r} (x - \zeta^l). \tag{5.2.34}$$

Putting $x = 1$, we have

$$\begin{aligned} p = 2r+1 &= \prod_{l=1}^{2r} (1-\zeta^l) \\ &= \prod_{k=1}^{r} (1-\zeta^k) \cdot (1-\zeta^{-k}) \\ &= \prod_{k=1}^{r} (1-\zeta^k) \cdot \zeta^{-k} \cdot (\zeta^k - 1) \\ &= \prod_{k=1}^{r} (-\zeta^{-k}) \cdot \prod_{k=1}^{r} (1-\zeta^k)^2 \\ &= (-1)^r \cdot \zeta^{-\frac{r(r+1)}{2}} \cdot \prod_{k=1}^{r} (1-\zeta^k)^2. \end{aligned} \tag{5.2.35}$$

□

Let $\epsilon := (-1)^r \cdot \zeta^{-\frac{r(r+1)}{2}} \in \mathcal{O}^\times$. Note that for any pair of integers j, k such that $j \not\equiv 0 \mod p$ and $k \not\equiv 0 \mod p$, we have

$$\frac{1-\zeta^j}{1-\zeta^k} \in \mathcal{O}^\times. \tag{5.2.36}$$

This is because, in $\mathbb{Z}/p\mathbb{Z}$, we can write j as a multiple of k, so the quotient in (5.2.36) can be written as a sum of elements in $\mathcal{O}$, and is hence in $\mathcal{O}$. Conversely, we can write k as a multiple of j, then the inverse of the quotient in (5.2.36) is also a sum of elements in $\mathcal{O}$, hence in $\mathcal{O}$.

Combining Lemma 5.2.2 and the above observation, we have the following corollary.

Corollary 5.2.1.

$$\sqrt{p} = u \prod_{k=1}^{r} (1-\theta_k), \tag{5.2.37}$$

for some $u \in \mathcal{O}^\times$.

Proof. Note first that the two square roots of ϵ are in $\mathcal{O}^\times$. Indeed, since 4 is invertible in $\mathbb{Z}/p\mathbb{Z}$, $\zeta^{\frac{-r(r+1)}{4}}$ is well-defined in $\mathcal{O}^\times$. Moreover, since $(i^r)^2 = (-1)^r$, we have that the two square roots of ϵ are

$$\pm\, i^r \cdot \zeta^{\frac{-r(r+1)}{4}}, \tag{5.2.38}$$

both of which are in $\mathcal{O}^\times$.

By Lemma 5.2.2, there exists a choice of square root of ϵ (depending on p), denoted by $\mathfrak{e}$, such that

$$\sqrt{p} = \mathfrak{e} \prod_{k=1}^{r} (1-\zeta^k) \tag{5.2.39}$$

The discussion above shows that $\mathfrak{e} \in \mathcal{O}^\times$.

We also have, for any $1 \le k \le 2r$,

$$\eta_k := \frac{1-\zeta^k}{1-\theta_k} = \frac{1-\zeta^k}{1-\zeta^{rk^2}} \in \mathcal{O}^\times. \tag{5.2.40}$$

Setting $u = \mathfrak{e}\prod_{k=1}^{r}\eta_k$, we have

$$\sqrt{p} = \mathfrak{e}\prod_{k=1}^{r}(1-\zeta^k) = \prod_{k=1}^{r}(1-\theta_k)\cdot\left(\mathfrak{e}\prod_{k=1}^{r}\eta_k\right) = u\prod_{k=1}^{r}(1-\theta_k). \tag{5.2.41}$$

Note that u, as product of elements in $\mathcal{O}^\times$, is in $\mathcal{O}^\times$. □

Proposition 5.2.3. *The 0-th column of $V^{-1}\boldsymbol{s}V$ is a vector in $\mathcal{O}^{r+1}$.*

Proof. By Proposition 5.2.2, we have, for any $j = 0, ..., r$,

$$(V^{-1}\boldsymbol{s}V)_{j0} = \sum_{l=0}^{r}(V^{-1})_{jl}(\boldsymbol{s}V)_{l0} = (V^{-1})_{j0}\cdot\sqrt{p}. \tag{5.2.42}$$

To prove the proposition, we have to show that $(V^{-1})_{j0}\cdot\sqrt{p} \in \mathcal{O}$.

Note that

$$\sum_{j=0}^{r} V_{kj}\cdot(V^{-1})_{j0} = \sum_{j=0}^{r}\theta_k^j\cdot(V^{-1})_{j0} = \delta_{k,0}. \tag{5.2.43}$$

We now consider the polynomial

$$P_0(x) = \sum_{j=0}^{r}(V^{-1})_{j0}\cdot x^j. \tag{5.2.44}$$

By Equation (5.2.43) we have

$$\begin{aligned} &P_0(\theta_0) = 1, \\ &P_0(\theta_k) = 0,\ \forall k = 1, ..., r. \end{aligned} \tag{5.2.45}$$

Therefore, by the Lagrangian interpolation formula, we have

$$P_0(x) = \sum_{j=0}^{r}(V^{-1})_{j0} \cdot x^j = \prod_{n=1}^{r} \frac{x-\theta_n}{1-\theta_n}. \tag{5.2.46}$$

By comparing coefficients, we can write down explicit formulas for $(V^{-1})_{j0}$. More importantly, we observe that

$$(V^{-1})_{j0} \cdot \prod_{n=1}^{r}(1-\theta_n) \in \mathcal{O}, \tag{5.2.47}$$

since Equation (5.2.46) implies that $(V^{-1})_{j0} \cdot \prod_{n=1}^{r}(1-\theta_n)$ is the coefficient of the integral polynomial $\prod_{n=1}^{r}(x-\theta_n)$. By Corollary 5.2.1 , we have for the unit $u \in \mathcal{O}^\times$

$$(V^{-1})_{j0} \cdot \sqrt{p} = (V^{-1})_{j0} \cdot u \prod_{n=1}^{r}(1-\theta_n) \in \mathcal{O}. \tag{5.2.48}$$

□

By Proposition 5.2.3, we are left to show that the l-th column vector of $\boldsymbol{s}V$ for $1 \leq l \leq r+1$ and all the column vectors of $\boldsymbol{t}V$ have the property that after multiplying them V^{-1} from left we get vectors in $\mathcal{O}^{r+1}$.

In light of Proposition 5.2.2, we have the following observation:

Lemma 5.2.3. *The l-th column vector of $\boldsymbol{s}V$ for $1 \leq l \leq r$ and all the column vectors of $\boldsymbol{t}V$ are, up to a scalar multiplication by $\pm i$ or ± 1, of the form $[1, {\theta_1}^j, \theta_2^j, \cdots, \theta_r^j]^t$ for some $0 \leq j \leq 2r$.*

Proof. This is a direct result of Proposition 5.2.2 and the definition of $\boldsymbol{t}$. □

Consider the polynomial $h(x) = (x-1)(x-\theta_1)\cdots(x-\theta_r) = x^{r+1}+b_1x^r+\cdots+b_{r+1}$ and its companion matrix

$$C_h = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -b_{r+1} & -b_r & -b_{r-1} & -b_{r-2} & \cdots & -b_1 \end{pmatrix}. \tag{5.2.49}$$

Note that for all $k = 1, ..., r+1$, $b_k \in \mathcal{O}$. Hence, $C_h \sqsubset \mathcal{O}$.

Proposition 5.2.4.

$$V^{-1}\boldsymbol{t}V = (C_h)^t. \tag{5.2.50}$$

In particular, $V^{-1}\boldsymbol{t}V \sqsubset \mathcal{O}$.

Proof. As discuss before (see the paragraph below Equation (5.2.19)), we have distinct eigenvalues $\theta_0, ..., \theta_r$ of C_h. For $j = 0, 1, ..., r$, the eigenvector of C_h corresponding to the eigenvalue θ_j is exactly the j-th column vector of V. Therefore, V^t diagonalizes C_h:

$$(V^t)^{-1}C_h(V^t) = \boldsymbol{t}. \tag{5.2.51}$$

Taking the transpose of both sides, we have

$$V(C_h)^t(V^{-1}) = \boldsymbol{t}, \tag{5.2.52}$$

as $\boldsymbol{t}$ is diagonal. Thus

$$V^{-1}\boldsymbol{t}V = (C_h)^t. \tag{5.2.53}$$

□

It is immediate from the above proposition that for any $0 \le j \le 2r$, $V^{-1}\boldsymbol{t}^j V \sqsubset \mathcal{O}$.

By Proposition 5.2.4 to prove Proposition 5.2.1 it remains to show that $V^{-1}\boldsymbol{s}V \sqsubset \mathcal{O}$.

Proof of Proposition 5.2.1 . For $0 \le j \le 2r$, the column vector $[1, {\theta_1}^j, \theta_2^j, \cdots, \theta_r^j]^t$ described in Proposition 5.2.3 is exactly the first column of $\boldsymbol{t}^j V$. Therefore, by the above observation, $V^{-1} \cdot [1, {\theta_1}^j, \theta_2^j, \cdots, \theta_r^j]^t \sqsubset \mathcal{O}$. By Proposition 5.2.3 , all column vectors of $V^{-1}\boldsymbol{s}V$ are in the form of $V^{-1} \cdot [1, {\theta_1}^k, \theta_2^k, \cdots, \theta_r^k]^t$ for some $0 \le k \le 2r$, so $V^{-1}\boldsymbol{s}V \sqsubset \mathcal{O}$. $\square$

We will denote the integral matrices $W^{-1}\mathfrak{s}W$ and $W^{-1}\mathfrak{t}W$ by $\mathcal{S}^{\mathbb{1}}$ and $\mathcal{T}^{\mathbb{1}}$ respectively.

5.3 Z-specialization

As noted in Section 3.3.4, $\rho_{1,1}^Z$ factors through a $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$ representation on $\mathbb{V}_{1,1}^Z$ since $\theta_Z = 1$.

According to the fusion rules the standard basis for $\mathbb{V}_{1,1}^Z$ is $\{v^Z(Y_1), ..., v^Z(Y_r)\}$, and in order to compute the S^Z and T^Z actions with respect to this basis we start again with the auxiliary matrix $\tilde{S}^Z$.

5.3.1 The auxiliary matrix $\tilde{S}^Z$

In this section we compute the auxiliary matrix $\tilde{S}^Z$ according to Equation (4.2.13). By symmetry, we only need to compute $\tilde{S}_{Y_jY_k}^{Y_{[2k]}}$ for $1 \le k \le j \le r$. And we have to discuss the cases when $j > k$ and when $j = k$ separately.

(Y_j, Y_k)-entry, $j > k$:

Recall the fusion rule when $j > k$:

$$Y_j \otimes Y_k \cong Y_{j-k} \oplus Y_{[j+k]}. \tag{5.3.1}$$

Furthermore, if $j + k \leq r$, we have $[j+k] - (j-k) = 2k > 0$. Otherwise, $[j+k] - (j-k) = (p-j-k) - j + k = p - 2j \geq p - 2r = 1$. So for any $1 \leq k < j \leq r$, we have $j - k < [j+k]$.

Following the indexing conventions in [AFT16] Section (4.2.2), we write

$$F^{Y_k}_{Y_jY_jY_k} = \begin{pmatrix} \left(F^{Y_k}_{Y_jY_jY_k}\right)_{\mathbb{1}Y_{j-k}} & \left(F^{Y_k}_{Y_jY_jY_k}\right)_{\mathbb{1}Y_{[j+k]}} \\ \left(F^{Y_k}_{Y_jY_jY_k}\right)_{ZY_{j-k}} & \left(F^{Y_k}_{Y_jY_jY_k}\right)_{ZY_{[j+k]}} \end{pmatrix} \tag{5.3.2}$$

and

$$G^{Y_k}_{Y_jY_jY_k} = \begin{pmatrix} \left(G^{Y_k}_{Y_jY_jY_k}\right)_{Y_{j-k}\mathbb{1}} & \left(G^{Y_k}_{Y_jY_jY_k}\right)_{Y_{j-k}Z} \\ \left(G^{Y_k}_{Y_jY_jY_k}\right)_{Y_{[j+k]}\mathbb{1}} & \left(G^{Y_k}_{Y_jY_jY_k}\right)_{Y_{[j+k]}Z} \end{pmatrix}. \tag{5.3.3}$$

The values of the entries in the above matrices are given in [AFT16] as

$$F^{Y_k}_{Y_jY_jY_k} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ (-1)^{(j-k+1)} & (-1)^{(j-k)} \end{pmatrix} \tag{5.3.4}$$

and

$$G^{Y_k}_{Y_jY_jY_k} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & (-1)^{(j-k+1)} \\ 1 & (-1)^{(j-k)} \end{pmatrix}. \tag{5.3.5}$$

In particular, we record the following values:

$$\begin{aligned}
\left(F^{Y_k}_{Y_jY_jY_k}\right)_{\mathbb{1}Y_{j-k}} &= \frac{1}{\sqrt{2}} \\
\left(F^{Y_k}_{Y_jY_jY_k}\right)_{\mathbb{1}Y_{[j+k]}} &= \frac{1}{\sqrt{2}} \\
\left(G^{Y_k}_{Y_jY_jY_k}\right)_{Y_{j-k}Z} &= \frac{(-1)^{(j-k+1)}}{\sqrt{2}} \\
\left(G^{Y_k}_{Y_jY_jY_k}\right)_{Y_{[j+k]}Z} &= \frac{(-1)^{(j-k)}}{\sqrt{2}}.
\end{aligned} \tag{5.3.6}$$

By Equation (4.2.20), the data involving R-matrices is readily computed:

$$\begin{aligned}
R^{Y_{j-k}}_{Y_jY_k} \cdot R^{Y_{j-k}}_{Y_kY_j} &= \exp\left(2(h_{Y_{j-k}} - h_{Y_j} - h_{Y_k})\pi i\right) \\
&= \zeta^{r\left((j-k)^2-j^2-k^2\right)} = \zeta^{-2rjk} = \zeta^{jk},
\end{aligned} \tag{5.3.7}$$

using that $\zeta^p = \zeta^{2r+1} = 1$.

Similarly,

$$\begin{aligned}
R^{Y_{[j+k]}}_{Y_jY_k} \cdot R^{Y_{[j+k]}}_{Y_kY_j} &= \exp\left(2(h_{Y_{[j+k]}} - h_{Y_j} - h_{Y_k})\pi i\right) \\
&= \zeta^{r\left((j+k)^2-j^2-k^2\right)} = \zeta^{2rjk} = \zeta^{-jk}.
\end{aligned} \tag{5.3.8}$$

Therefore,

$$\begin{aligned}
\tilde{S}^{Z}_{Y_jY_k} &= R^{Y_{j-k}}_{Y_jY_k} \cdot R^{Y_{j-k}}_{Y_kY_j} \cdot \left(F^{Y_k}_{Y_jY_jY_k}\right)_{\mathbb{1}Y_{j-k}} \cdot \left(G^{Y_k}_{Y_jY_jY_k}\right)_{Y_{j-k}Z} \\
&\quad + R^{Y_{[j+k]}}_{Y_jY_k} \cdot R^{Y_{[j+k]}}_{Y_kY_j} \cdot \left(F^{Y_k}_{Y_jY_jY_k}\right)_{\mathbb{1}Y_{[j+k]}} \cdot \left(G^{Y_k}_{Y_jY_jY_k}\right)_{Y_{[j+k]}Z} \\
&= \zeta^{jk} \cdot \frac{1}{\sqrt{2}} \cdot \frac{(-1)^{(j-k+1)}}{\sqrt{2}} + \zeta^{-jk} \cdot \frac{1}{\sqrt{2}} \cdot \frac{(-1)^{(j-k)}}{\sqrt{2}} \\
&= \frac{(-1)^{(j+k+1)}}{2} \cdot \left(\zeta^{jk} - \zeta^{-jk}\right).
\end{aligned} \tag{5.3.9}$$

Note that changing the power of (-1) from $(j-k+1)$ to $(j+k+1)$ does not change the value, but we prefer this notation because it reminds us that $\tilde{S}^{Z}_{Y_jY_k}$ is symmetric in j and k.

(Y_k, Y_k)**-entry:**

For $1 \leq k \leq r$, we already have the data we need. Namely, from Equations (5.1.3), (5.1.4), (5.1.5) and (5.1.6) we have

$$\begin{aligned}
\tilde{S}^Z_{Y_kY_k} = & \left(R^{\mathbb{1}}_{Y_kY_k}\right)^2 \cdot \left(F^{Y_k}_{Y_kY_kY_k}\right)_{\mathbb{1}\mathbb{1}} \cdot \left(G^{Y_k}_{Y_kY_kY_k}\right)_{\mathbb{1}Z} \\
& + \left(R^{Z}_{Y_kY_k}\right)^2 \cdot \left(F^{Y_k}_{Y_kY_kY_k}\right)_{\mathbb{1}Z} \cdot \left(G^{Y_k}_{Y_kY_kY_k}\right)_{ZZ} \\
& + \left(R^{Y_{[2k]}}_{Y_kY_k}\right)^2 \cdot \left(F^{Y_k}_{Y_kY_kY_k}\right)_{\mathbb{1}Y_{[2k]}} \cdot \left(G^{Y_k}_{Y_kY_kY_k}\right)_{Y_{[2k]}Z} \\
= & \, \zeta^{k^2} \cdot \frac{1}{2} \cdot \frac{(-1)}{2} + \zeta^{k^2} \cdot \frac{(-1)}{2} \cdot \frac{1}{2} + \zeta^{-k^2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \\
= & \, -\frac{1}{2}\left(\zeta^{k^2} - \zeta^{-k^2}\right).
\end{aligned} \tag{5.3.10}$$

Since $(-1)^{(k+k+1)} = -1$, we can combine Equations (5.3.9) and (5.3.10) to get

$$\tilde{S}^Z_{Y_jY_k} = \frac{(-1)^{(j+k+1)}}{2} \cdot \left(\zeta^{jk} - \zeta^{-jk}\right) \tag{5.3.11}$$

for any $1 \leq j, k \leq r$.

5.3.2 The action of S^Z

By Equation (4.2.14), for any $b, c \in \{Y_1, ..., Y_r\}$,

$$S^Z_{cb} = \tilde{S}^Z_{bc} \cdot \frac{d_c^2 \cdot (F^c_{ccc})_{Z\mathbb{1}} \left(F^Z_{ZZZ}\right)_{\mathbb{1}\mathbb{1}}}{2\sqrt{p}\,(F^c_{ccc})_{\mathbb{1}\mathbb{1}}}. \tag{5.3.12}$$

First, recall that $d_{Y_k} = 2$ for any $k = 1, ..., r$. As in Equation (5.1.3), we have $\left(F^{Y_k}_{Y_kY_kY_k}\right)_{Z\mathbb{1}} = -1/2$ and $\left(F^{Y_k}_{Y_kY_kY_k}\right)_{\mathbb{1}\mathbb{1}} = 1$ for any $k = 1, ..., r$. Finally, by [AFT16], $\left(F^Z_{ZZZ}\right)_{\mathbb{1}\mathbb{1}} = 1$. Therefore, we have

$$S^Z_{cb} = \tilde{S}^Z_{bc} \cdot \frac{4 \cdot \left(\frac{-1}{2}\right)}{2\sqrt{p} \cdot \left(\frac{1}{2}\right)} = \tilde{S}^Z_{bc} \cdot \frac{(-2)}{\sqrt{p}}. \tag{5.3.13}$$

Plugging Equation (5.3.11) into the above equation, we have

$$\begin{aligned} S^Z_{Y_j Y_k} &= \frac{(-1)^{(j+k+1)}}{2} \cdot \left(\zeta^{jk} - \zeta^{-jk}\right) \cdot \frac{(-2)}{\sqrt{p}} \\ &= \frac{(-1)^{(j+k)}}{\sqrt{p}} \cdot \left(\zeta^{jk} - \zeta^{-jk}\right). \end{aligned} \tag{5.3.14}$$

5.3.3 The action of T^Z

By Theorem 3.3.2 , we have the following proposition.

Proposition 5.3.1. *With respect to the standard basis,*

$$T^Z = \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_r \end{pmatrix}. \tag{5.3.15}$$

5.3.4 Integral basis for $\mathbb{V}^Z_{1,1}$

In this section we prove the following theorem.

Theorem 5.3.1. *There is a basis of $\mathbb{V}^Z_{1,1}$ such that, with respect to this basis, the matrix presentation of S^Z and T^Z are integral.*

Before proceeding to the proof, we have to recall some basic facts about the so-called V_p-TQFT associated to p [BHMV95, GMvW04] in order to use the integrality result from [GMvW04]. In the literature, the V_p-theory is also referred to as the SO(3)-theory associated to p. However, in order to avoid confusion with the $\mathrm{SO}(p)_2$ modular category in this paper, we prefer using the term V_p-theory. An extensive discussion on the V_p-theory is beyond the scope of the paper. The interested reader is referred to [BHMV95] for details.

In this dissertation, we focus on the modular data of the modular category $\mathcal{L}_p$ that gives rise to the V_p-theory. According to [BHMV95], the modular category $\mathcal{L}_p$ has r simple objects. Hence the S-matrix, denoted by $\mathfrak{s}_{V_p}$, is of size $r \times r$, and its entries are given by

$$\left(\mathfrak{s}_{V_p}\right)_{jk} = (-1)^{j+k} \cdot \frac{\zeta - \zeta^{-1}}{i\sqrt{p}} \cdot \frac{\zeta^{jk} - \zeta^{-jk}}{\zeta - \zeta^{-1}} = \frac{(-1)^{j+k}}{i\sqrt{p}} \cdot \left(\zeta^{jk} - \zeta^{-jk}\right) \tag{5.3.16}$$

for $j, k = 1, ..., r$.

Recall that $q = \exp\left(\frac{1}{p}\pi i\right)$. The T-matrix of the V_p-theory is given by

$$\left(\mathfrak{t}_{V_p}\right)_{jk} = \delta_{j,k} \cdot (-1)^{j-1} \cdot q^{j^2-1} \tag{5.3.17}$$

for $j, k = 1, ..., r$.

The relation between the modular data of the V_p-theory and the mapping class group action on $\mathbb{V}^Z_{1,1}$ is as follows.

Theorem 5.3.2.

$$\begin{aligned} S^Z &= i \cdot \mathfrak{s}_{V_p}, \\ T^Z &= \zeta^r \cdot \left(\mathfrak{t}_{V_p}\right)^{-1}. \end{aligned} \tag{5.3.18}$$

Proof. With $i^{-1} = i^3$, the statement on $\mathfrak{s}_{V_p}$ and S^Z is the consequence of direct comparison between Equation (5.3.14) and Equation (5.3.16).

The statement about $\mathfrak{t}_{V_p}$ and T^Z is based on the observation that

$$q = \exp\left(\frac{1}{p}\pi i\right) = \exp\left(\frac{p-2r}{p}\pi i\right) = -\exp\left(\frac{-2r}{p}\pi i\right) = -\zeta^{-r}. \tag{5.3.19}$$

Consequently, for any $j = 1, ..., r$,

$$\begin{aligned} \left(\mathfrak{t}_{V_p}\right)_{jj} &= (-1)^{j-1} \cdot q^{j^2-1} = (-1)^{j-1} \cdot (-\zeta^{-r})^{j^2-1} \\ &= (-1)^{j^2+j-2} \cdot \zeta^r \cdot \zeta^{-rj^2} = \zeta^r \cdot \theta_j^{-1} \\ &= \zeta^r \cdot \left(T^Z\right)^{-1}_{jj}. \end{aligned} \tag{5.3.20}$$

Note that $(-1)^{j^2+j-2} = (-1)^{j^2+j} = (-1)^{j(j+1)} = 1$. □

Consider the matrix $\mathfrak{Z} = (\mathfrak{Z}_{jk})$, whose entries are given by

$$\mathfrak{Z}_{jk} = (-1)^j \cdot \zeta^{-rk(j^2-1)} \cdot \left(\zeta^j - \zeta^{-j}\right) \tag{5.3.21}$$

for $j, k = 1, ..., r$. In [GMvW04, Theorem 6.1], it is shown that

$$\mathfrak{Z}^{-1}\mathfrak{s}_{V_p}\mathfrak{Z} \sqsubset \mathcal{O} \tag{5.3.22}$$

and

$$\mathfrak{Z}^{-1}\mathfrak{t}_{V_p}\mathfrak{Z} \sqsubset \mathcal{O}. \tag{5.3.23}$$

We can now return to the proof of Theorem 5.3.1 .

Proof of Theorem 5.3.1 . It is enough to show that

$$\mathfrak{Z}^{-1}S^Z\mathfrak{Z} \sqsubset \mathcal{O} \tag{5.3.24}$$

and

$$\mathfrak{z}^{-1}T^Z\mathfrak{z} \sqsubset \mathcal{O}. \tag{5.3.25}$$

By Theorem 5.3.2 ,

$$\mathfrak{z}^{-1}S^Z\mathfrak{z} = i \cdot \mathfrak{z}^{-1}\mathfrak{s}_{V_p}\mathfrak{z} \sqsubset \mathcal{O} \tag{5.3.26}$$

since $i \in \mathcal{O}$.

By Theorem 5.3.2 , we have

$$\mathfrak{z}^{-1}\left(T^Z\right)^{-1}\mathfrak{z} = \zeta^{-r} \cdot \mathfrak{z}^{-1}\mathfrak{t}_{V_p}\mathfrak{z} \sqsubset \mathcal{O} \tag{5.3.27}$$

as $\zeta^{-r} \in \mathcal{O}$. We can tweak this result once we realize that the diagonal entries of T^Z are all p-th roots of unity. Hence for any $j = 1, ..., r$,

$$\left(T^Z\right)_{jj} = \theta_j = (\theta_j)^{1-p} = (\theta_j)^{-2r} = \left(\left(T^Z\right)^{-1}_{jj}\right)^{2r}. \tag{5.3.28}$$

Since T^Z is diagonal, this means

$$T^Z = \left(\left(T^Z\right)^{-1}\right)^{2r}, \tag{5.3.29}$$

and, consequently,

$$\mathfrak{z}^{-1}T^Z\mathfrak{z} = \mathfrak{z}^{-1}\left(\left(T^Z\right)^{-1}\right)^{2r}\mathfrak{z} = \left(\mathfrak{z}^{-1}\left(T^Z\right)^{-1}\mathfrak{z}\right)^{2r} \sqsubset \mathcal{O}. \tag{5.3.30}$$

$\square$

We will denote $\mathfrak{z}^{-1}S^Z\mathfrak{z}$ and $\mathfrak{z}^{-1}T^Z\mathfrak{z}$ by $\mathcal{S}^Z$ and $\mathcal{T}^Z$ respectively.

Chapter 6

Discussion

By the Main Theorem, for any $a = \mathbb{1}, Z, Y_1, ..., Y_r$, with respect to the integral bases $\mathcal{B}^a$, the matrix presentation of any element in $\rho^a_{1,1}(\widetilde{\Gamma_{1,1}})$ is integral. For $a = \mathbb{1}, Z, Y_1, ..., Y_r$, recall that (Equation (5.0.1))

$$n_a = \dim(\mathbb{V}^a_{1,1}), \tag{6.0.1}$$

and that $\Lambda_a = \mathrm{span}_{\mathcal{O}}(\mathcal{B}^a)$ is the $\widetilde{\Gamma_{1,1}}$-invariant, full-rank, free $\mathcal{O}$-lattice in $\mathbb{V}^a_{1,1}$. We have a representation

$$\begin{aligned} \mathcal{Q}^a : \widetilde{\Gamma_{1,1}} &\to \mathrm{GL}(\Lambda_a), \\ \overline{S}_1 &\mapsto \overline{\xi}(\mathrm{SO}(p)_2) \cdot \mathcal{S}^a \\ T_1 &\mapsto \mathcal{T}^a \\ K &\mapsto \overline{\xi}(\mathrm{SO}(p)_2) \cdot \mathrm{Id}_{n_a} \end{aligned} \tag{6.0.2}$$

where $\mathrm{GL}(\Lambda_a)$ denotes the group of $\mathcal{O}$-linear automorphisms of Λ_a. Consider the ring homomorphism $\mathbb{Z}[\zeta, i] \to (\mathbb{Z}/p\mathbb{Z})[i]$ sending ζ to 1. It induces a group homomorphism

$$\mathfrak{P} : \mathrm{GL}(\Lambda_a) \longrightarrow \mathrm{GL}((\mathbb{Z}/p\mathbb{Z}[i])^{n_a}), \tag{6.0.3}$$

note that $(\mathbb{Z}/p\mathbb{Z}[i])^{n_a} \cong \Lambda_a/(\zeta - 1)\Lambda_a$.

Consider the composition

$$\widetilde{\Gamma_{1,1}} \xrightarrow{\mathcal{Q}^a} \mathrm{GL}(\Lambda_a) \xrightarrow{\mathfrak{P}} \mathrm{GL}((\mathbb{Z}/p\mathbb{Z}[i])^{n_a}). \tag{6.0.4}$$

It is interesting to study the image of $\widetilde{\Gamma_{1,1}}$ under the reduction map $\mathfrak{P} \circ \mathcal{Q}^a$ since the reductions may have connections to topological information of 3-manifolds, such as the Casson-Lescop invariants and the Milnor torsion, see for example, [Ker02]. We will also call $\mathfrak{P} \circ \mathcal{Q}^a(\gamma)$ the *reduction* of γ for any $\gamma \in \widetilde{\Gamma_{1,1}}$. Recall that $\overline{\xi}(\mathrm{SO}(p)_2) = i^{3r} \in \mathcal{O}$ by Equation (5.0.4). The reduction of the central generator $K \in \widetilde{\Gamma_{1,1}}$ is readily seen to be

$$\mathfrak{P}(\mathcal{Q}^a(K)) = i^{3r} \cdot \mathrm{Id}_{n_a} \tag{6.0.5}$$

for any $a = \mathbb{1}, Z, Y_1, ..., Y_r$.

In the first section, we will give examples of explicit expressions of the integral matrices $\mathcal{S}^a$ and $\mathcal{T}^a$ for $a \in \{\mathbb{1}, Z, Y_1, ..., Y_r\}$. For any odd prime $p = 2r + 1$ and any $a \in \{Y_1, ..., Y_r\} \subset \mathrm{Irr}\,(\mathrm{SO}(p)_2)$, we give explicitly $\mathcal{S}^a$ and $\mathcal{T}^a$ for each a. We then give, the integral matrices $\mathcal{S}^a$ and $\mathcal{T}^a$ for $a \in \{\mathbb{1}, Z\} \subset \mathrm{Irr}\,(\mathrm{SO}(p)_2)$ for p = 3, 5 and 7. We will also give the reductions of $\overline{S}_1$ and T_1 using the integral matrices provided in each case.

The rest of the chapter focuses on other properties of the representations $\rho^a_{1,1}$. We will show that the image of every representation of the mapping class group specialized to $a \in \mathrm{Irr}\,(\mathrm{SO}(p)_2)$ is a finite subgroup of $\mathrm{GL}(\mathbb{V}^a_{1,1})$. We will also discuss the relationship between $\rho^{\mathbb{1}}_{1,1}$ and $\rho^Z_{1,1}$ and the Weil representations over the finite field $\mathbb{Z}/p\mathbb{Z}$.

6.1 Examples of the integral matrices and their reductions

For any odd prime p and any $k = 1, ..., r$, we have explicit expressions for $\mathcal{S}^{Y_{[2k]}}$ and $\mathcal{T}^{Y_{[2k]}}$ given by Equations (5.1.51) and (5.1.55). We have

$$\mathfrak{P}(\mathcal{Q}^{Y_{[2k]}}(\overline{S}_1)) = i^{3r}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.1.1}$$

and

$$\mathfrak{P}(\mathcal{Q}^{Y_{[2k]}}(T_1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i^r & 0 \end{pmatrix}. \tag{6.1.2}$$

Next, we consider the $\mathbb{1}$-specialization. Since $\mathbb{V}^{\mathbb{1}}_{1,1}$ decomposes into a direct sum of $\widetilde{\Gamma_{1,1}}$-invariant subspaces $H_1 \oplus H_2$, we have that $\Lambda_{\mathbb{1}}$ also decomposes into $\widetilde{\Gamma_{1,1}}$-invariant sublattices. Let $\Lambda^1_{\mathbb{1}} := H_1 \cap \Lambda_{\mathbb{1}}$ and let $\Lambda^2_{\mathbb{1}} := H_2 \cap \Lambda_{\mathbb{1}}$. It is enough to give the restrictions of $\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(\overline{S}_1))$ and $\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(T_1))$ to $\Lambda^1_{\mathbb{1}}$ and $\Lambda^2_{\mathbb{1}}$. By Equations (5.2.12) and (5.2.13), we have

$$\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(\overline{S}_1)|_{\Lambda^1_{\mathbb{1}}}) = i^{3r}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.1.3}$$

and

$$\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(T_1)|_{\Lambda^1_{\mathbb{1}}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i^r & 0 \end{pmatrix}. \tag{6.1.4}$$

Remark 6.1.1. If we further reduce i to 1, we recover the $\Gamma_{1,1}$ action on the space of functions on quadratic forms on $H_1(\Sigma_{1,0}, \mathbb{Z}/2\mathbb{Z})$ with Arf invariant 0, or the Birman-Craggs homomorphisms via the identifications given by Johnson. The interested reader is referred to [Wri92, Joh80, BC78].

By Proposition 5.2.4 , $\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}} = V^{-1}\boldsymbol{t}V$ (see Equations (5.2.19) and (5.2.18)) can be given by the transpose of the companion matrix of the polynomial $\prod_{j=0}^{r}(x - \theta_j)$. Therefore, $\mathfrak{P}(\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}})$ is the transpose of the companion matrix of the polynomial $\prod_{j=0}^{r}(x - 1) = (x - 1)^{r+1}$, since the map $\mathbb{Z}[\zeta, i] \to (\mathbb{Z}/p\mathbb{Z})[i]$ sending ζ to 1 is a ring homomorphism. Hence, we have

$$\begin{aligned}&\left(\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(T_1)|_{\Lambda^2_{\mathbb{1}}})\right)_{jk} = \left(\mathfrak{P}(\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}})\right)_{jk}\\ &= \begin{cases} \delta_{j,k+1} & \text{if } 0 \le j \le r \text{ and } 0 \le k \le (r-1) \\ (-1)^{r-j} \cdot \binom{r+1}{j} \mod p & \text{if } 0 \le j \le r \text{ and } k = r \end{cases}.\end{aligned} \tag{6.1.5}$$

In the rest of this section, we will give explicitly $\mathcal{S}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}$, $\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}$, $\mathcal{S}^Z$ and $\mathcal{T}^Z$ for the categories $\mathrm{SO}(3)_2$, $\mathrm{SO}(5)_2$ and $\mathrm{SO}(7)_2$. We will also give the corresponding reductions $\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(\overline{S}_1)|_{\Lambda^2_{\mathbb{1}}})$, $\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(T_1)|_{\Lambda^2_{\mathbb{1}}})$, $\mathfrak{P}(\mathcal{Q}^Z(\overline{S}_1))$ and $\mathfrak{P}(\mathcal{Q}^Z(T_1))$. Recall that by our convention of notations, $\mathcal{S}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}} = V^{-1}\boldsymbol{s}V$, $\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}} = V^{-1}\boldsymbol{t}V$, $\mathcal{S}^Z = \mathfrak{Z}^{-1}S^Z\mathfrak{Z}$ and $\mathcal{T}^Z = \mathfrak{Z}^{-1}T^Z\mathfrak{Z}$ where the explicit expressions of $\boldsymbol{s}$, $\boldsymbol{t}$, V, S^Z, T^Z and $\mathfrak{Z}$ can be find in Equations (5.2.17), (5.2.18), (5.2.19), (5.3.14), (5.3.15), and (5.3.21), respectively.

6.1.1 $\mathrm{SO}(3)_2$

In this section, $p = 3$, $r = 1$, $\zeta = \zeta_3 = \exp\left(\frac{2}{3}\pi i\right)$, and $\theta_j = \zeta_3^{j^2}$ for $j = 0, 1$.

We start with the $\mathbb{1}$-specialization. As is shown in Proposition 5.2.4 , $\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}$ is the transpose of the companion matrix of the polynomial $\prod_{j=0}^{1}(x - \theta_j) = (x - 1)(x - \zeta_3)$.

We have

$$\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}} = \begin{pmatrix} 0 & -\zeta_3 \\ 1 & 1+\zeta_3 \end{pmatrix} \tag{6.1.6}$$

and by direct computation, we have

$$\mathcal{S}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}} = -i\zeta_3 \cdot \begin{pmatrix} -\zeta_3 & 1 \\ 1 & \zeta_3 \end{pmatrix}. \tag{6.1.7}$$

Therefore, we have

$$\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(T_1)|_{\Lambda^2_{\mathbb{1}}}) = \mathfrak{P}(\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \tag{6.1.8}$$

and

$$\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(\overline{S}_1)|_{\Lambda^2_{\mathbb{1}}}) = \mathfrak{P}(i^3 \cdot \mathcal{S}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}) = (-i) \cdot (-i) \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}. \tag{6.1.9}$$

Note that by Lemma 5.0.1 , $n_Z = 1$. By direct computation, we have

$$\mathcal{T}^Z = (\zeta_3), \tag{6.1.10}$$

and

$$\mathcal{S}^Z = (i). \tag{6.1.11}$$

Hence we have

$$\mathfrak{P}(\mathcal{Q}^Z(T_1)) = \mathfrak{P}(\mathcal{T}^Z) = (1) \tag{6.1.12}$$

and

$$\mathfrak{P}(\mathcal{Q}^Z(\overline{S}_1)) = \mathfrak{P}(i^3 \cdot \mathcal{S}^Z) = (1)\,. \tag{6.1.13}$$

6.1.2 $\mathrm{SO}(5)_2$

In this section, $p = 5$, $r = 2$, $\zeta = \zeta_5 = \exp\left(\frac{2}{5}\pi i\right)$, and $\theta_j = \zeta_5^{2j^2}$ for $j = 0, 1, 2$.

We start with the $\mathbb{1}$-specialization. By Proposition 5.2.4, the transpose of the companion matrix of the polynomial $\prod_{j=0}^{2}(x-\theta_j) = (x-1)(x-\zeta_5^2)(x-\zeta_5^3)$ gives $\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}$. So we have

$$\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \frac{1}{2}(\sqrt{5}-1) \\ 0 & 1 & \frac{1}{2}(1-\sqrt{5}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \zeta_5+\zeta_5^4 \\ 0 & 1 & -(\zeta_5+\zeta_5^4) \end{pmatrix}. \tag{6.1.14}$$

By direct computation, we have

$$\mathcal{S}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}} = \begin{pmatrix} \frac{1}{2}(\sqrt{5}-1) & \frac{1}{2}(\sqrt{5}-1) & 0 \\ 1 & \frac{1}{2}(1-\sqrt{5}) & 0 \\ \frac{1}{2}(\sqrt{5}-1) & -1 & 1 \end{pmatrix} = \begin{pmatrix} \zeta_5+\zeta_5^4 & \zeta_5+\zeta_5^4 & 0 \\ 1 & -(\zeta_5+\zeta_5^4) & 0 \\ \zeta_5+\zeta_5^4 & -1 & 1 \end{pmatrix}. \tag{6.1.15}$$

Therefore, we have

$$\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(T_1)|_{\Lambda^2_{\mathbb{1}}}) = \mathfrak{P}(\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}. \tag{6.1.16}$$

and

$$\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(\overline{S}_1)|_{\Lambda^2_{\mathbb{1}}}) = \mathfrak{P}(i^6 \cdot \mathcal{S}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}) = (-1) \cdot \begin{pmatrix} 2 & 2 & 0 \\ 1 & -2 & 0 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 0 \\ 4 & 2 & 0 \\ 3 & 1 & 4 \end{pmatrix}. \tag{6.1.17}$$

We directly compute the matrices given by the Z-specialization. We have

$$\mathcal{T}^Z = \begin{pmatrix} \zeta_5^2 + \zeta_5^3 & \zeta_5^2 \\ -\zeta_5^3 & 0 \end{pmatrix} \tag{6.1.18}$$

and

$$\mathcal{S}^Z = \begin{pmatrix} -(1+\zeta_5^4) & \zeta_5^2 - \zeta_5^4 \\ -(\zeta_5^2+\zeta_5^3) & 1+\zeta_5^4 \end{pmatrix}. \tag{6.1.19}$$

Therefore, we have

$$\mathfrak{P}(\mathcal{Q}^Z(T_1)) = \mathfrak{P}(\mathcal{T}^Z) = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} \tag{6.1.20}$$

and

$$\mathfrak{P}(\mathcal{Q}^Z(\overline{S}_1)) = \mathfrak{P}(i^6 \cdot \mathcal{S}^Z) = (-1) \cdot \begin{pmatrix} -2 & 0 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}. \tag{6.1.21}$$

6.1.3 SO(7)$_2$

In this section, $p = 7$, $r = 3$, $\zeta = \zeta_7 = \exp\left(\frac{2}{7}\pi i\right)$, and $\theta_j = \zeta_7^{3j^2}$ for $j = 0, 1, 2, 3$.

We start with the $\mathbb{1}$-specialization. By Proposition 5.2.4, the transpose of the companion matrix of the polynomial $\prod_{j=0}^{3}(x - \theta_j) = (x-1)(x-\zeta_7^3)(x-\zeta_7^5)(x-\zeta_7^6)$ gives $\mathcal{T}^{\mathbb{1}}|_{\Lambda_{\mathbb{1}}^2}$. So we have

$$\mathcal{T}^{\mathbb{1}}|_{\Lambda_{\mathbb{1}}^2} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \frac{1}{2}\left(1 + i\sqrt{7}\right) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2}\left(1 - i\sqrt{7}\right) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -(\zeta_7^3 + \zeta_7^5 + \zeta_7^6) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -(\zeta_7 + \zeta_7^2 + \zeta_7^4) \end{pmatrix} \tag{6.1.22}$$

and

$$\mathcal{S}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}} = i \cdot \begin{pmatrix} 1 & -\frac{1}{2}\left(1+i\sqrt{7}\right) & 0 & 0 \\ \frac{1}{2}\left(1-i\sqrt{7}\right) & -1 & 0 & 0 \\ -\frac{1}{2}\left(1+i\sqrt{7}\right) & -\frac{1}{2}\left(1-i\sqrt{7}\right) & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \tag{6.1.23}$$

$$= i \cdot \begin{pmatrix} 1 & \zeta_7^3+\zeta_7^5+\zeta_7^6 & 0 & 0 \\ -(\zeta_7+\zeta_7^2+\zeta_7^4) & -1 & 0 & 0 \\ \zeta_7^3+\zeta_7^5+\zeta_7^6 & \zeta_7+\zeta_7^2+\zeta_7^4 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}.$$

Hence, we have

$$\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(T_1)|_{\Lambda^2_{\mathbb{1}}}) = \mathfrak{P}(\mathcal{T}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}) = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{6.1.24}$$

and

$$\begin{aligned} &\mathfrak{P}(\mathcal{Q}^{\mathbb{1}}(\overline{S}_1)|_{\Lambda^2_{\mathbb{1}}}) = \mathfrak{P}(i^9 \cdot \mathcal{S}^{\mathbb{1}}|_{\Lambda^2_{\mathbb{1}}}) \\ =& i \cdot i \cdot \begin{pmatrix} 1 & 3 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ 3 & 3 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 4 & 4 & 0 & 1 \\ 1 & 6 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{6.1.25}$$

For the Z-specialization, we have

$$\mathcal{T}^Z = \begin{pmatrix} \zeta_7^3+\zeta_7^5+\zeta_7^6 & \zeta_7^3 & 0 \\ -(\zeta_7+\zeta_7^5+\zeta_7^6) & 0 & \zeta_7^3 \\ \zeta_7 & 0 & 0 \end{pmatrix} \tag{6.1.26}$$

and

$$\mathcal{S}^Z = i \cdot \begin{pmatrix} \zeta_7^3 & \zeta_7^2-\zeta_7^5 & \zeta_7^6-\zeta_7^5 \\ 1 & 1-\zeta_7^2 & -(\zeta_7^2+\zeta_7^6) \\ -(1+\zeta_7^2+\zeta_7^3) & \zeta_7^2+\zeta_7^4+\zeta_7^5 & \zeta_7^2-\zeta_7^3 \end{pmatrix}. \tag{6.1.27}$$

Therefore, we have

$$\mathfrak{P}(\mathcal{Q}^Z(T_1)) = \mathfrak{P}(\mathcal{T}^Z) = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.1.28}$$

and

$$\mathfrak{P}(\mathcal{Q}^Z(\overline{S}_1)) = \mathfrak{P}(i^9 \cdot \mathcal{S}^Z) = i \cdot i \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -2 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 6 & 0 & 2 \\ 3 & 4 & 0 \end{pmatrix}. \tag{6.1.29}$$

6.2 Image finiteness

In this section, we will apply [NS10, Theorem 7.1] to prove the following theorem.

Theorem 6.2.1. $\rho^a_{1,1}(\widetilde{\Gamma_{1,1}})$ *is a finite subgroup of* $\mathrm{GL}(\mathbb{V}^a_{1,1})$ *for any* $a \in \{\mathbb{1}, Z, Y_1, ..., Y_r\}$.

Proof. Recall that $n_a = \dim(\mathbb{V}^a_{1,1})$ (Equation (5.0.1)). It is enough to show that

$\langle \mathcal{S}^a, \mathcal{T}^a, K^a, \zeta \cdot \mathrm{Id}_{n_a} \rangle$, the subgroup generated by $\mathcal{S}^a, \mathcal{T}^a, K^a, \zeta \cdot \mathrm{Id}_{n_a}$ in $\mathrm{GL}(\mathbb{V}^a_{1,1})$, is finite. As $K^a = i^{3r} \cdot \mathrm{Id}_{n_a}$ (cf. Theorem 3.3.3 , Lemma 5.0.2), it is enough to show that $\langle \mathcal{S}^a, \mathcal{T}^a, i \cdot \mathrm{Id}_{n_a}, \zeta \cdot \mathrm{Id}_{n_a} \rangle$. Since $i \cdot \mathrm{Id}_{n_a}$, $\zeta \cdot \mathrm{Id}_{n_a}$ are central and have finite order, it is enough to show that $\langle \mathcal{S}^a, \mathcal{T}^a \rangle$ is finite. By Theorem 6.2.2 , $\langle \mathcal{S}^a, \mathcal{T}^a \rangle$ is finite for $a = Y_j$, $j = 1, ..., r$. By Theorem 6.2.3 , $\langle \mathcal{S}^a, \mathcal{T}^a \rangle$ is finite for $a = \mathbb{1}$. Finally, by Theorem 6.2.4 , $\langle \mathcal{S}^a, \mathcal{T}^a \rangle$ is finite for $a = Z$. □

6.2.1 The $Y_{[2k]}$-specialization

In this section, we prove

Theorem 6.2.2. *$\langle \mathcal{S}^{Y_{[2k]}}, \mathcal{T}^{Y_{[2k]}} \rangle$, the group generated by $\mathcal{S}^{Y_{[2k]}}$ and $\mathcal{T}^{Y_{[2k]}}$, is a finite subgroup of $\mathrm{GL}(\mathbb{V}^{Y_{[2k]}}_{1,1})$.*

Proof. Let S_3 be the permutation group on 3 elements. Let $Q := \langle i, \zeta \rangle \in \mathbb{C}^\times$ be the subgroup of non-zero elements in $\mathbb{C}$ generated by i and ζ. Consider the subset $\boldsymbol{F} \subset \mathrm{GL}(3, \mathbb{C})$ consisting of matrices $M \in \mathrm{GL}(3, \mathbb{C})$ satisfying the condition that there exist $\pi \in S_3$, and $q_1, q_2, q_3 \in Q$ such that for $j, k = 1, 2, 3$,

$$M_{jk} = \delta_{j,\pi(k)} \cdot q_j. \tag{6.2.1}$$

It is readily seen that $\boldsymbol{F}$ is a subgroup of $\mathrm{GL}(3, \mathbb{C})$, and $\boldsymbol{F} \cong S_3 \ltimes Q^3$. So $\boldsymbol{F}$ is a finite subgroup of $\mathrm{GL}(3, \mathbb{C})$.

By Equations (5.1.51) and (5.1.55), $\langle \mathcal{S}^{Y_{[2k]}}, \mathcal{T}^{Y_{[2k]}} \rangle \subset \boldsymbol{F}$, so $\langle \mathcal{S}^{Y_{[2k]}}, \mathcal{T}^{Y_{[2k]}} \rangle$ is finite. □

6.2.2 The $\mathbb{1}$- and Z-speciaizations

Let $\mathcal{C}$ be a modular category, and let $\mathfrak{s}_{\mathcal{C}}$ and $\mathfrak{t}_{\mathcal{C}}$ be its modular data. Let $\hat{\sigma}$ and $\hat{\tau}$ be generators of $\mathrm{SL}(2, \mathbb{Z})$ (see Equations (2.2.15) and (2.2.16)). Let $\iota_{\mathcal{C}}$ be a cube root of $\overline{\xi}(\mathcal{C})$. In other words, $\iota_{\mathcal{C}}$ satisfies

$$\iota_{\mathcal{C}}^3 = \overline{\xi}(\mathcal{C}). \tag{6.2.2}$$

It is shown in [ENO05, Corollary 8.18.3] that $\overline{\xi}(\mathcal{C})$ is a root of unity, hence $\iota_{\mathcal{C}}$ is also a root of unity. By [BK01, Corollary 3.1.8], the homomorphism

$$\begin{aligned} \tilde{\mu}_{\mathcal{C}} : \mathrm{SL}(2,\mathbb{Z}) &\longrightarrow \mathrm{GL}(K_0(\mathcal{C}) \otimes \mathbb{C}) \\ \hat{\sigma} &\mapsto \mathfrak{s}_{\mathcal{C}} \\ \hat{\tau} &\mapsto \iota_{\mathcal{C}} \cdot \mathfrak{t}_{\mathcal{C}} \end{aligned} \tag{6.2.3}$$

is a linear representation of $\mathrm{SL}(2,\mathbb{Z})$. Moreover, $\tilde{\mu}_{\mathcal{C}}$ lifts the projective representation $\mu_{\mathcal{C}}$ defined in Equation (2.2.17) in the following sense. Let $\mathfrak{p} : \mathrm{GL}(K_0(\mathcal{C}) \otimes \mathbb{C}) \to \mathrm{PGL}(K_0(\mathcal{C}) \otimes \mathbb{C})$ be the natural projection. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{Z}) & \xrightarrow{\tilde{\mu}_{\mathcal{C}}} & \mathrm{GL}(K_0(\mathcal{C}) \otimes \mathbb{C}) \\ & {\scriptstyle \mu_{\mathcal{C}}} \searrow \quad \swarrow {\scriptstyle \mathfrak{p}} & \\ & \mathrm{PGL}(K_0(\mathcal{C}) \otimes \mathbb{C}) & \end{array} . \tag{6.2.4}$$

It is shown in [NS10, Theorem 7.1] that for any modular category $\mathcal{C}$, the image $\tilde{\mu}_{\mathcal{C}}(\mathrm{SL}(2,\mathbb{Z}))$ is a finite subgroup of $\mathrm{GL}(K_0(\mathcal{C}) \otimes \mathbb{C})$. In other words, $|\langle \mathfrak{s}_{\mathcal{C}}, \iota_{\mathcal{C}} \cdot \mathfrak{t}_{\mathcal{C}} \rangle| < \infty$.

Recall that the modular data of $\mathrm{SO}(p)_2$ is denoted by $\mathfrak{s}$ and $\mathfrak{t}$. Let $\iota_{\mathrm{SO}(p)_2}$ be a fixed cube root of $\overline{\xi}(\mathrm{SO}(p)_2) = i^{3r}$. Note that $\iota_{\mathrm{SO}(p)_2}$ is a root of unity. Now we are ready to establish the following theorems.

Theorem 6.2.3. *$\langle \mathcal{S}^{\mathbb{1}}, \mathcal{T}^{\mathbb{1}} \rangle$ is a finite subgroup of $\mathrm{GL}(\mathbb{V}_{1,1}^{\mathbb{1}})$.*

Proof. By the definition of $\mathcal{S}^{\mathbb{1}}$ and $\mathcal{T}^{\mathbb{1}}$ at the end of Section 5.2 , we have

$$|\langle \mathcal{S}^{\mathbb{1}},\ \mathcal{T}^{\mathbb{1}} \rangle| = |\langle W^{-1}\mathfrak{s}W,\ W^{-1}\mathfrak{t}W \rangle| = |\langle \mathfrak{s},\ \mathfrak{t} \rangle|, \tag{6.2.5}$$

because conjugation is a group automorphism. Moreover, we have

$$\langle \mathfrak{s},\ \mathfrak{t}\rangle \subset \langle \mathfrak{s},\ \mathfrak{t},\ \iota_{\mathrm{SO}(p)_2}\cdot \mathrm{Id}\rangle = \langle \mathfrak{s},\ \iota_{\mathrm{SO}(p)_2}\cdot \mathfrak{t},\ \iota_{\mathrm{SO}(p)_2}\cdot \mathrm{Id}\rangle, \tag{6.2.6}$$

since $\iota_{\mathrm{SO}(p)_2}\cdot \mathrm{Id}$ is central in $\mathrm{GL}(\mathbb{V}^a_{1,1})$.

Since $\iota_{\mathrm{SO}(p)_2}$ is a root of unity, $\langle \iota_{\mathrm{SO}(p)_2}\cdot \mathrm{Id}\rangle$, the subgroup of $\mathrm{GL}(\mathbb{V}^{\mathbb{1}}_{1,1})$ generated by $\iota_{\mathrm{SO}(p)_2}\cdot \mathrm{Id}$, is finite. In other words, $|\langle \iota_{\mathrm{SO}(p)_2}\cdot \mathrm{Id}\rangle| < \infty$. Therefore,

$$\begin{aligned} |\langle \mathcal{S}^{\mathbb{1}},\ \mathcal{T}^{\mathbb{1}}\rangle| = |\langle \mathfrak{s},\ \mathfrak{t}\rangle| &\le |\langle \mathfrak{s},\ \iota_{\mathrm{SO}(p)_2}\cdot \mathfrak{t},\ \iota_{\mathrm{SO}(p)_2}\cdot \mathrm{Id}\rangle| \\ &\le |\langle \mathfrak{s},\ \iota_{\mathrm{SO}(p)_2}\cdot \mathfrak{t}\rangle|\cdot|\langle \iota_{\mathrm{SO}(p)_2}\cdot \mathrm{Id}\rangle| < \infty. \end{aligned} \tag{6.2.7}$$

□

Theorem 6.2.4. $\langle \mathcal{S}^Z, \mathcal{T}^Z\rangle$ *is a finite subgroup of* $\mathrm{GL}(\mathbb{V}^Z_{1,1})$.

Proof. By definition and property of conjugation, we have

$$|\langle \mathcal{S}^Z,\ \mathcal{T}^Z\rangle| = |\langle \mathfrak{Z}^{-1}S^Z\mathfrak{Z},\ \mathfrak{Z}^{-1}T^Z\mathfrak{Z}\rangle| = |\langle S^Z,\ T^Z\rangle|. \tag{6.2.8}$$

Note that ι_{V_p} is a root of unity by the argument at the beginning of this section. By Theorem 5.3.2 , and a similar argument in the proof of Theorem 6.2.3 , we have

$$\begin{aligned} |\langle S^Z, T^Z\rangle| &= |\langle i\cdot \mathfrak{s}_{V_p},\ \zeta^{-r}\cdot \mathfrak{t}_{V_p}\rangle| \\ &\le |\langle \mathfrak{s}_{V_p},\ \iota_{V_p}\cdot \mathfrak{t}_{V_p},\ \iota_{V_p}\cdot \mathrm{Id},\ i\cdot \mathrm{Id},\ \zeta\cdot \mathrm{Id}\rangle| \\ &\le |\langle \mathfrak{s}_{V_p},\ \iota_{V_p}\cdot \mathfrak{t}_{V_p}\rangle|\cdot|\langle \iota_{V_p}\cdot \mathrm{Id}\rangle|\cdot|\langle i\cdot \mathrm{Id}\rangle|\cdot|\langle \zeta\cdot \mathrm{Id}\rangle| < \infty. \end{aligned} \tag{6.2.9}$$

□

6.3 Relationship to the Weil representation

In this section, we relate $\rho_{1,1}^{\mathbb{1}}$ and $\rho_{1,1}^{Z}$ to the Weil representation over the finite field $\mathbb{Z}/p\mathbb{Z}$. There is a vast amount of research on the Weil representations over various fields, for example, see [Wei64, Gér77, LV80]. In this section, we will only extract some essential ingredients of the representation of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ following [Cha12].

Our starting point is the observation that both $\rho_{1,1}^{\mathbb{1}}$ and $\rho_{1,1}^{Z}$ give rise to projective representations of $\mathrm{SL}(2,\mathbb{Z})$. More precisely, let

$$\hat{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; \hat{\tau} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6.3.1}$$

be the generators of $\mathrm{SL}(2,\mathbb{Z})$. As is mentioned at the end of Section 2.2.2, for any finite dimensional complex vector space V, and any automorphism $M \in \mathrm{GL}(V)$ of V, we denote the equivalence class of M in the group $\mathrm{PGL}(V)$ by $\{M\}$.

Direct computation shows that the map

$$\begin{aligned} \mu^{\mathbb{1}} : \mathrm{SL}(2,\mathbb{Z}) &\longrightarrow \mathrm{PGL}(\mathbb{V}_{1,1}^{\mathbb{1}}) \\ \hat{\sigma} &\mapsto \{U^{-1}\mathfrak{s}U\} \\ \hat{\tau} &\mapsto \{U^{-1}\mathfrak{t}U\} \end{aligned} \tag{6.3.2}$$

is a group homomorphism. Recall that we denote $\mathbb{V}_{1,1}^{\mathbb{1}}$ by H. Recall that by Lemma 5.2.1, we have a decomposition of H into $\widetilde{\Gamma_{1,1}}$-invariant subspaces $H \cong H_1 \oplus H_2$. Then we have the following group homomorphism

$$\begin{aligned} \mu_2^{\mathbb{1}} : \mathrm{SL}(2,\mathbb{Z}) &\longrightarrow \mathrm{PGL}(H_2) \\ Q &\mapsto \{\mu^{\mathbb{1}}(Q)|_{H_2}\}. \end{aligned} \tag{6.3.3}$$

Similarly, direct computation shows that the map

$$\begin{aligned} \mu^Z : \mathrm{SL}(2,\mathbb{Z}) &\longrightarrow \mathrm{PGL}(\mathbb{V}^Z_{1,1}) \\ \hat{\sigma} &\mapsto \{S^Z\} \\ \hat{\tau} &\mapsto \{T^Z\} \end{aligned} \tag{6.3.4}$$

is a group homomorphism.

To construct the Weil representation, we first define the *Heisenberg group* $\mathcal{H}_p$ by

$$\mathcal{H}_p = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{Z}/p\mathbb{Z} \right\}. \tag{6.3.5}$$

$\mathcal{H}_p$ is a non-abelian finite group of order p^3. Multiplication in $\mathcal{H}_p$ is matrix multiplication. There is an embedding of $\mathbb{Z}/p\mathbb{Z}$ into $\mathcal{H}_p$ via

$$\begin{aligned} \mathcal{I} : \mathbb{Z}/p\mathbb{Z} &\hookrightarrow \mathcal{H}_p, \\ z &\mapsto \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{6.3.6}$$

which identifies $\mathbb{Z}/p\mathbb{Z}$ with $Z(\mathcal{H}_p)$, the center of $\mathcal{H}_p$. We also have a surjective homomorphism from $\mathcal{H}_p$ to $(\mathbb{Z}/p\mathbb{Z})^2$ given by

$$\begin{aligned} \mathcal{P} : \mathcal{H}_p &\twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^2, \\ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} &\mapsto \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned} \tag{6.3.7}$$

It is easy to check that $\mathcal{I}$ and $\mathcal{P}$ give rise to a short exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\mathcal{I}} \mathcal{H}_p \xrightarrow{\mathcal{P}} (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow 0. \tag{6.3.8}$$

Consider the action of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ on $(\mathbb{Z}/p\mathbb{Z})^2$ given by

$$\begin{aligned} &\mathcal{A} : \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{Aut}((\mathbb{Z}/p\mathbb{Z})^2) \\ &\mathcal{A}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}, \end{aligned} \tag{6.3.9}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ and $\begin{pmatrix} x \\ y \end{pmatrix} \in (\mathbb{Z}/p\mathbb{Z})^2$.

It is shown in [Cha12, Section 2.2] that with a suitable choice of section of $\mathcal{P}$, $\mathcal{A}$ can be lifted to $\mathcal{H}_p$, and the lifted action is trivial on the center $Z(\mathcal{H}_p)$.

Let $\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})$ be the space of complex-valued functions on $\mathbb{Z}/p\mathbb{Z}$. It is clear that $\dim((\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))) = p$.

For any non-trivial irreducible central character $\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$, we can define a representation $\pi_\varphi : \mathcal{H}_p \to \mathrm{GL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$ by

$$\left(\pi_\varphi\left(\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}\right)(f)\right)(a) = \varphi(-xa+z)f(a-y), \tag{6.3.10}$$

for any $\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{H}_p$ and $f \in \mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})$.

Since π_φ is a p-dimensional representation, by the representation theory of finite groups, it is either a direct sum of p 1-dimensional representations or irreducible. However, in the first case, $\pi_\varphi|_{Z(\mathcal{H}_p)}$ should be trivial, which contradicts to our assumption on φ.

By the Stone von-Neumann Theorem (for example, see [Pra09, Theorem 3.1]), if two irreducible representations of $\mathcal{H}_p$ coincide on the center $Z(\mathcal{H}_p)$, then they are equivalent. Now let φ be any nontrivial irreducible central character. For any $Q \in \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$, consider the representation $\pi_\varphi \circ Q$, a p-dimensional representation of $\mathcal{H}_p$ with the property

$$(\pi_\varphi \circ Q)|_{Z(\mathcal{H}_p)} = \varphi = \pi_\varphi|_{Z(\mathcal{H}_p)}. \tag{6.3.11}$$

By the same argument about the dimension of the representation theory of the finite group $\mathcal{H}_p$ as in the previous paragraph, we know that $\pi_\varphi \circ Q$ is also irreducible. Hence $\pi_\varphi \circ Q$ is equivalent to π_φ, in other words, there is an intertwining operator, denoted by $W_\varphi(Q) \in \mathrm{GL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$ such that the diagram

$$\begin{array}{ccc} \mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\pi_\varphi(h)} & \mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) \\ {\scriptstyle W_\varphi(Q)}\downarrow & & \downarrow{\scriptstyle W_\varphi(Q)} \\ \mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\pi_\varphi(Q(h))} & \mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) \end{array} \tag{6.3.12}$$

commutes for all $h \in \mathcal{H}_p$.

By Schur's lemma, $W_\varphi(Q)$ is unique up to scalar, so we get a projective representation of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$. More precisely, we have a group homomorphism

$$W_\varphi : \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{PGL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})). \tag{6.3.13}$$

This projective representation is called the *Weil representation* of $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$ (with respect to φ). For simplicity, by an abuse of notation, we will present an equivalence class in $\mathrm{PGL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$ by one of its representatives in $\mathrm{GL}(\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}))$.

Again, let $\hat{\sigma}$ and $\hat{\tau}$ be the generators of $\mathrm{SL}(2, \mathbb{Z})$. Their reductions mod p generate

$\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z})$. By an abuse of notation, we will not distinguish $\hat{\sigma}$ and $\hat{\tau}$ from their reductions. For $j \in \mathbb{Z}/p\mathbb{Z}$, let $f_j : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the function defined by

$$f_j(x) = \delta_{j,x},\ \forall x \in \mathbb{Z}/p\mathbb{Z}. \tag{6.3.14}$$

The set $\{f_j | j \in \mathbb{Z}/p\mathbb{Z}\}$ is a basis for $\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})$.

Let φ be a nontrivial central character of $\mathbb{Z}/p\mathbb{Z}$. To describe the Weil representation with respect to φ, it suffices to give the matrices of $W_\varphi(\hat{\sigma})$ and $W_\varphi(\hat{\tau})$ with respect to the basis given above. It is not difficult to compute that

$$W_\varphi(\hat{\sigma}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \varphi(1) & \varphi(2) & \cdots & \varphi(p-1) \\ 1 & \varphi(2) & \varphi(4) & \cdots & \varphi(2(p-1)) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \varphi(p-1) & \varphi(2(p-1)) & \cdots & \varphi((p-1)^2) \end{pmatrix} \tag{6.3.15}$$

and that

$$W_\varphi(\hat{\tau}) = \begin{pmatrix} 1 & & & & & \\ & \varphi(-\frac{1^2}{2}) & & & & \\ & & \varphi(-\frac{2^2}{2}) & & & \\ & & & \varphi(-\frac{3^2}{2}) & & \\ & & & & \ddots & \\ & & & & & \varphi(-\frac{(p-1)^2}{2}) \end{pmatrix}. \tag{6.3.16}$$

As before, $\frac{1}{2}$ is understood as the reciprocal of 2 in $\mathbb{Z}/p\mathbb{Z}$.

Note that W_φ is reducible. Indeed, it is easy to see that

$$E^{even} := \mathrm{span}_{\mathbb{C}}(\{f_k + f_{p-k} | k = 0, 1, \cdots, r\}) \tag{6.3.17}$$

and

$$E^{odd} := \mathrm{span}_{\mathbb{C}}(\{f_k - f_{p-k} | k = 1, \cdots, r\}) \tag{6.3.18}$$

are two invariant subspaces. Moreover, we have a decomposition of $\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z})$ into invariant subspaces $\mathcal{L}^2(\mathbb{Z}/p\mathbb{Z}) \cong E^{even} \oplus E^{odd}$.

By (6.3.15) and (6.3.16), we have

$$W_{\varphi}^{even}(\hat{\sigma}) = W_{\varphi}(\hat{\sigma})|_{E^{even}} = \begin{pmatrix} 1 & \boldsymbol{a}^t \\ 2 \cdot \boldsymbol{a} & B \end{pmatrix}. \tag{6.3.19}$$

Here, $\boldsymbol{a}$ is as in Equation (4.2.57), and B is an $r \times r$-matrix with entries given by

$$B_{jk} = \varphi(jk) + \varphi(-jk), \ \forall j, k = 1, \cdots, r. \tag{6.3.20}$$

We also have

$$W_{\varphi}^{even}(\hat{\tau}) = W_{\varphi}(\hat{\tau})|_{E^{even}} = \begin{pmatrix} 1 & & & & \\ & \varphi(-\frac{1^2}{2}) & & & \\ & & \varphi(-\frac{2^2}{2}) & & \\ & & & \ddots & \\ & & & & \varphi(-\frac{r^2}{2}) \end{pmatrix}. \tag{6.3.21}$$

Let $\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ be the central character given by

$$\varphi(j) = \zeta^j, \ \forall j \in \mathbb{Z}/p\mathbb{Z}. \tag{6.3.22}$$

We have $\sqrt{p}A = B$, where A is defined in Equation (4.2.58). We also have

$$2r \equiv -1 \mod p \;\Rightarrow\; r \equiv -\frac{1}{2} \mod p \;\Rightarrow\; \varphi(-\frac{j^2}{2}) = \zeta^{rj^2} = \theta_j. \tag{6.3.23}$$

Therefore, we have

$$W_{\varphi}^{even}(\hat{\sigma}) = \sqrt{p}\begin{pmatrix} \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot \boldsymbol{a}^t \\ \frac{2}{\sqrt{p}} \cdot \boldsymbol{a} & A \end{pmatrix}, \tag{6.3.24}$$

and

$$W_{\varphi}^{even}(\hat{\tau}) = \begin{pmatrix} 1 & & & \\ & \theta_1 & & \\ & & \ddots & \\ & & & \theta_r \end{pmatrix}. \tag{6.3.25}$$

Define the linear map

$$\begin{aligned} \mathfrak{h} : H_2 &\longrightarrow E^{even} \\ \mathbb{1} + Z &\mapsto 2f_0 \\ Y_j &\mapsto f_j + f_{p-j},\ \forall j = 1, ..., r. \end{aligned} \tag{6.3.26}$$

It is easy to see that $\mathfrak{h}$ is an isomorphism. Therefore, it induces an isomorphism $\mathfrak{h}^* : \mathrm{PGL}(H_2) \xrightarrow{\cong} \mathrm{PGL}(E^{even})$.

Let

$$\mathrm{mod}_p : \mathrm{SL}(2, \mathbb{Z}) \longrightarrow \mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z}) \tag{6.3.27}$$

be the reduction mod p homomorphism.

Theorem 6.3.1. *Let φ be the central character defined in Equation (6.3.22). Then $\mu_2^{\mathbb{1}}$ factors through W_φ^{even}. More precisely, we have the following commutative diagram*

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{Z}) & \xrightarrow{\mu_2^{\mathbb{1}}} & \mathrm{PGL}(H_2) \\ \downarrow{\scriptstyle \mathrm{mod}_p} & & \cong \downarrow{\scriptstyle \mathfrak{h}^*} \\ \mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{W_\varphi^{even}} & \mathrm{PGL}(E^{even}) \end{array} . \tag{6.3.28}$$

Proof. By Equations (5.2.14), (5.2.15), (6.3.24) and (6.3.25), we know that $\mathfrak{h}^* \circ \mu_2^{\mathbb{1}}$ and $W_\varphi^{even} \circ$mod are only different by a scalar multiple, hence as projective representations, they are the same. □

We can also determine the odd part of the Weil representation with respect to the basis $\{f_k - f_{p-k} | k = 0, 1, \cdots, r\}$. For $j, k = 1, ..., r$,

$$\left(W_\varphi^{odd}(\hat{\sigma})\right)_{jk} = (W_\varphi(\hat{\sigma})|_{E^{odd}})_{jk} = \varphi(jk) - \varphi(-jk) \tag{6.3.29}$$

and

$$\left(W_\varphi^{odd}(\hat{\tau})\right)_{jk} = (W_\varphi(\hat{\tau})|_{E^{odd}})_{jk} = \delta_{j,k} \cdot \varphi(-\frac{j^2}{2}). \tag{6.3.30}$$

If we define again φ as in Equation (6.3.22), we can rewrite the above equations as

$$\left(W_\varphi^{odd}(\hat{\sigma})\right)_{jk} = \zeta^{jk} - \zeta^{-jk} \tag{6.3.31}$$

and

$$\left(W_\varphi^{odd}(\hat{\tau})\right)_{jk} = \delta_{j,k} \cdot \zeta^{-\frac{j^2}{2}} = \delta_{j,k} \cdot \zeta^{rj^2} = \delta_{j,k} \cdot \theta_j. \tag{6.3.32}$$

Consider the linear map

$$
\begin{aligned}
\mathfrak{u} : \mathbb{V}^Z_{1,1} &\longrightarrow E^{odd} \\
v^Z(Y_j) &\mapsto (-1)^j(f_j - f_{p-j}), \ \forall j = 1, ..., r.
\end{aligned}
\tag{6.3.33}
$$

It is easy to show that this is a linear isomorphism. Therefore, it induces an isomorphism of groups $\mathfrak{u}^* : \mathrm{PGL}(\mathbb{V}^Z_{1,1}) \xrightarrow{\cong} \mathrm{PGL}(E^{odd})$. More precisely, for any $\{F\} \in \mathrm{PGL}(\mathbb{V}^Z_{1,1})$, we have $\mathfrak{u}^*(\{F\}) = \{\mathfrak{u} \circ F \circ \mathfrak{u}^{-1}\}$.

Theorem 6.3.2. *Let φ be the central character defined in Equation (6.3.22), then μ^Z factors through W^{odd}_φ. More precisely, we have the following commutative diagram*

$$
\begin{array}{ccc}
\mathrm{SL}(2,\mathbb{Z}) & \xrightarrow{\mu^Z} & \mathrm{PGL}(\mathbb{V}^Z_{1,1}) \\
\Big\downarrow{\scriptstyle \mathrm{mod}_p} & & {\scriptstyle\cong}\Big\downarrow{\scriptstyle \mathfrak{u}^*} \\
\mathrm{SL}(2,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{W^{odd}_\varphi} & \mathrm{PGL}(E^{odd})
\end{array}
\quad . \tag{6.3.34}
$$

Proof. By Equations (5.3.14), (5.3.15), and the definition of $\mathfrak{u}^*$, we have

$$
\begin{aligned}
\left(\mathfrak{u}^*(\mu^Z(\hat{\sigma}))\right)_{jk} &= (-1)^{j+k} \cdot \left((\mu^Z(\hat{\sigma}))\right)_{jk} \\
&= (-1)^{j+k} \cdot \frac{(-1)^{j+k}}{\sqrt{p}} \cdot (\zeta^{jk} - \zeta^{-jk}) \\
&= \frac{1}{\sqrt{p}} \cdot \left(W^{odd}_\varphi(\hat{\sigma})\right)_{jk},
\end{aligned}
\tag{6.3.35}
$$

and

$$
\begin{aligned}
\left(\mathfrak{u}^*(\mu^Z(\hat{\tau}))\right)_{jk} &= (-1)^{j+k} \cdot \delta_{j,k}\theta_j \\
&= \delta_{j,k}\theta_j \\
&= \left(W^{odd}_\varphi(\hat{\tau})\right)_{jk}.
\end{aligned}
\tag{6.3.36}
$$

Note that for any pair of integers $j, k \in \mathbb{Z}$, $(-1)^{j+k} \cdot \delta_{j,k} = 1$ if $j = k$, $(-1)^{j+k} \cdot \delta_{j,k} = 0$ if $j \neq k$. So $(-1)^{j+k} \cdot \delta_{j,k} = \delta_{j,k}$. □

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