# ON THE POSITIVITY CONJECTURE: A DIGEST OF MASON'S COUNTEREXAMPLE

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## 1. Introduction

In this note, we give an explicit construction of a counterexample to the positivity conjecture on the second Frobenius-Schur indicators of a modular category. The positivity conjecture can be formulated in the following way [Wan10, Conjecture 4.26]:

Let $\mathcal{C}$ be a modular category, and let $X, Y$ be simple objects of $\mathcal{C}$. Then $N_{X,X^*}^{Y} > 0$ implies $\nu_2(Y) = 1$.

We will borrow ideas from Mason's preprint [Mas17], but our example comes from a smaller group than that in [Mas17]. In Section 3, we give the explicit character data of the counterexample implemented in `GAP`.

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## 2. Construction of a counterexample

Let $Q := Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ be the quaternion group. It is easy to check that the assignment $\alpha : Q \to \mathrm{GL}(2,3)$, given on generators by

$$a \mapsto \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},\ b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

embeds $Q$ into $\mathrm{GL}(2,3)$ as a subgroup. Therefore, $Q$ acts faithfully on $H := (\mathbb{Z}/3\mathbb{Z})^2$ by (left) matrix multiplication (with column vectors). For any $q \in Q$ and any $h \in H$, we denote this action by $q \cdot h$.

Let $G := H \rtimes_\alpha Q$ be the semidirect product of $H$ and $Q$ with respect to $\alpha$. As subgroups of $G$, $Q$ acts on $H$ by conjugation (in $G$). Note that $G$ is of order 72. Let $\lambda : H \to \mathbb{C}^\times$ be a character of $H$ defined by

$$\lambda(x, y) := \omega^{y-x}$$

for any $(x, y) \in H = (\mathbb{Z}/3\mathbb{Z})^2$, where $\omega = \exp(\frac{2\pi i}{3})$. Let $\chi := \mathrm{ind}_H^G(\lambda)$ be the representation of $G$ induced by $\lambda$.

**Proposition 1.** *The representation $\chi$ is irreducible.*

*Proof.* We prove the irreducibility of $\chi$ by computing its character. By the character formula of induced representations ([Ser77, Theorem 12]), for any $g \in G$, we have

$$\mathrm{char}(\chi)(g) = \sum_{\substack{q \in Q \\ q^{-1}gq \in H}} \lambda(q \cdot g) = \begin{cases} 0, & \text{if } g \notin H \\ -1, & \text{if } g \in H \setminus \{e_G\} \\ 8, & \text{if } g = e_G \end{cases}$$

where $e_G$ stands for the identity of $G$.

Therefore, the character inner product (following notations in [Ser77]) of $\chi$ with itself is given by

$$(\mathrm{char}(\chi)|\,\mathrm{char}(\chi)) = \frac{1}{72}\sum_{g\in G}|\,\mathrm{char}(\chi)|^2 = \frac{1}{72}\times(8^2+(-1)^2\times 8) = 1.$$

Hence, by [Ser77, Theorem 3], $\chi$ is irreducible. □

**Remark 1.** *We can also apply Mason's idea to give an alternative proof of the irreducibility of $\chi$, which goes as follows. By a corollary of Mackey's irreducibility criterion of induced representations (*[Ser77, Corollary 7.4.23]*), $\chi$ is irreducible if and only if*

*(1) $\lambda$ is irreducible;*

*(2) ${}^g\lambda \neq \lambda$ for every $g \notin H$.*

*Here, ${}^g\lambda : H \to \mathbb{C}^\times$ is defined by ${}^g\lambda(h) := \lambda(g^{-1}hg)$ for any $h \in H$. It is clear that $\lambda$ is irreducible, so it remains to prove (2), which is the result of direct computation using the expression for $\lambda$.*

We proceed with the following known facts. Firstly, $Q$ has four 1-dimensional (denoted by $\gamma_1, ..., \gamma_4$) and one 2-dimensional irreducible representation (denoted by $\phi$). Moreover, direct computation shows that $\nu_2(\phi) = -1$. In addition, by dimension counting, $\chi$ is the unique 8-dimensional representation of $G$ with $\nu_2(\chi) = 1$.

Since $\mathrm{Rep}(Q)$ is a braided (symmetric) spherical fusion full subcategory of $\mathrm{Rep}(G)$, any irreducible representation of $Q$ can be viewed as an irreducible representation of $G$ by pre-composing the quotient map $G \twoheadrightarrow Q$. In addition, for any $X \in \mathrm{Rep}(Q)$, its dimension and its second Frobenius-Schur indicator $\nu_2(X)$ computed in $\mathrm{Rep}(Q)$ is the same as $\nu_2(X)$ computed in $\mathrm{Rep}(G)$. More precisely, let $\xi \in \mathrm{Rep}(Q)$, by definition, $\nu_2(\xi)$ in $\mathrm{Rep}(Q)$ and in $\mathrm{Rep}(G)$ are given respectively as

$$\nu_2(\xi)_{\mathrm{Rep}(Q)} = \frac{1}{|Q|}\sum_{q\in Q}\mathrm{char}(\xi)(q^2),$$

and

$$\nu_2(\xi)_{\mathrm{Rep}(G)} = \frac{1}{|G|}\sum_{g\in G}\mathrm{char}(\xi)(g^2).$$

By definition, when we view $\xi$ as in $\mathrm{Rep}(G)$, we have $\xi(gh) = \xi(g)$ for any $g \in G$ and $h \in H$. Therefore, by the fact that $Q \cong G/H$, we have

$$\nu_2(\xi)_{\mathrm{Rep}(G)} = \frac{1}{|G|}\sum_{g\in G/H}\mathrm{char}(\xi)(g^2)|H| = \frac{1}{|Q|}\sum_{q\in Q}\mathrm{char}(\xi)(q^2) = \nu_2(\xi)_{\mathrm{Rep}(Q)}.$$

Similar argument holds for $\mathrm{Rep}(D(G))$. More precisely, since both $\mathrm{Rep}(Q)$ and $\mathrm{Rep}(G)$ are braided (symmetric) spherical fusion full subcategory of $\mathrm{Rep}(D(G))$, for any $X \in \mathrm{Rep}(Q)$, and for any $Y \in \mathrm{Rep}(G)$, we have

$$\nu_2(X)_{\mathrm{Rep}(Q)} = \nu_2(X)_{\mathrm{Rep}(D(G))}$$

and

$$\nu_2(Y)_{\mathrm{Rep}(G)} = \nu_2(Y)_{\mathrm{Rep}(D(G))}.$$

In particular, we have

$$\nu_2(\phi)_{\mathrm{Rep}(D(G))} = \nu_2(\phi)_{\mathrm{Rep}(Q)} = -1. \tag{1}$$

Let $\rho_G$ be the regular representation of $G$. It is standard that

$$\rho_G = \bigoplus_{j=1}^{4} \gamma_j \oplus 2\phi \oplus 8\chi.$$

**Theorem 1.** *The 2-dimensional representation $\phi$ is a constituent of $\chi \otimes \chi$ in* $\mathrm{Rep}(G)$.

*Proof.* It is well-known (or see Appendix) that

$$\rho_G \otimes \phi = \rho_G^{\oplus \deg(\phi)} = \rho_G \oplus \rho_G = 2\bigoplus_{j=1}^{4} \gamma_j \oplus 4\phi \oplus 16\chi. \tag{2}$$

Decomposing the left hand side, we have

$$\rho_G \otimes \phi = 4\phi \oplus 2\bigoplus_{j=1}^{4} \gamma_j \oplus (8\chi \otimes \phi),$$

where the first two summands are derived from the familiar representation theory of $Q$. Comparing both sides of Equation (2), we have

$$\chi \otimes \phi = 2\chi. \tag{3}$$

In other words, $\mathrm{Hom}_{\mathrm{Rep}(G)}(\chi\otimes\phi, \chi) \neq 0$, which implies that $\mathrm{Hom}_{\mathrm{Rep}(G)}(\chi\otimes\chi, \phi) \neq 0$, as both $\chi$ and $\phi$ are self-dual. □

As pointed out before, we can view $\chi$ and $\phi$ as objects in the modular category $\mathrm{Rep}(D(G))$. Since $\mathrm{Rep}(G)$ is a fusion full subcategory of $\mathrm{Rep}(D(G))$, we will still have $N_{\chi,\chi}^{\phi} = 2$ in $\mathrm{Rep}(D(G))$. Together with Equation (1) and Theorem 1 we have

**Theorem 2.** *In the modular category* $\mathrm{Rep}(D(G))$*, there exist irreducible representations $\chi, \phi \in \mathrm{Irr}(\mathrm{Rep}(D(G)))$ such that $N_{\chi,\chi^*}^{\phi} = N_{\chi,\chi}^{\phi} = 2$ and $\nu_2(\phi) = -1$.* □

The above theorem nullifies the positivity conjecture.

**Remark 2.** *Equation (3) implies that $\nu_2$ is not a fusion character. Indeed, by definition and the linearity of $\nu_2$, we have*

$$\nu_2(\chi \otimes \phi) = \nu_2(2\chi) = 2\nu_2(\chi) = 2,$$

*while*

$$\nu_2(\chi) \times \nu_2(\phi) = 1 \times (-1) = -1.$$

## 3. GAP implementation

In fact, we can identify $G$ with $\mathrm{PSU}(3,2)$ whose `GAP` ID is `SmallGroup(72, 41)`. We use the following code in `GAP` to get the information we need.

We first get the information of irreducible representations of $G$ by

```
G:=SmallGroup(72,41);;
Irr(G);
```

The output is

```
[Character(CharacterTable( <pc group of size 72 with 5 generators> ),
[ 1, 1, 1, 1, 1, 1 ]),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 1, -1, -1, 1, 1, 1 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 1, -1, 1, 1, 1, -1 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 1, 1, -1, 1, 1, -1 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 2, 0, 0, -2, 2, 0 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 8, 0, 0, 0, -1, 0 ] ) ]
```

We can see that among the 6 irreducible representations, there is a unique 8-dimensional representation, which is denoted by $\chi$ in the previous section.

Next, we compute the second Frobenius-Schur indicator of the above irreducible representations

```
Indicator(CharacterTable(G), 2);
```

The output is

```
[ 1, 1, 1, 1, -1, 1 ]
```

This means the 2-dimensional irreducible representation of $G$ has -1 as its $\nu_2$.

Finally, we decompose $\chi \otimes \chi$ into irreducible representations

```
ConstituentsOfCharacter(Irr(G)[6]*Irr(G)[6]);
```

The output is

```
[Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 1, -1, -1, 1, 1, 1 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 1, -1, 1, 1, 1, -1 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 1, 1, -1, 1, 1, -1 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 1, 1, 1, 1, 1, 1 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 2, 0, 0, -2, 2, 0 ] ),
Character( CharacterTable( <pc group of size 72 with 5 generators> ),
[ 8, 0, 0, 0, -1, 0 ] ) ]
```

We can see that the 2-dimensional irreducible representation of $G$ is indeed a constituent of $\chi \otimes \chi$.

## Appendix

For any braided spherical fusion category $\mathcal{C}$, let $\mathrm{Irr}(\mathcal{C})$ denote the set of isomorphism class of simple objects, and let $d_X$ be the categorical dimension of $X \in \mathbb{C}$. Let $R := \sum_{X \in \mathrm{Irr}(\mathcal{C})} d_X X$ be the regular element in the Grothendieck algebra of $\mathcal{C}$.

**Lemma 1.** *For any $V \in \mathcal{C}$, we have the equality in the Grothendieck algebra of $\mathcal{C}$*

$$RV = d_V R.$$

*Proof.*

$$
\begin{aligned}
RV &= \left( \sum_{X \in \mathrm{Irr}(\mathcal{C})} d_X X \right) V \\
&= \sum_{X \in \mathrm{Irr}(\mathcal{C})} d_X \sum_{Y \in \mathrm{Irr}(\mathcal{C})} N_{X,V}^{Y} Y \\
&= \sum_{Y \in \mathrm{Irr}(\mathcal{C})} Y \sum_{X \in \mathrm{Irr}(\mathcal{C})} N_{V,Y^*}^{X^*} d_{X^*} \\
&= \sum_{Y \in \mathrm{Irr}(\mathcal{C})} d_Y d_V Y \\
&= d_V R.
\end{aligned} \tag{4}
$$

□

## References


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