Hopf algebra and tensor categories

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Main Topics

- Definition & basic properties
- Comodules and Hopf modules
- ☆ Integral theory ⇝ (co)semisimplity
- Coradicals and filtrations
- Duality
- Drinfeld double
- Braided tensor categories

References

Montgomery. Hopf algebra and their actions on rings.

Sweedler. Hopf algebras.

Kassel, Quantum groups.

Radford. Hopf algebras.

Other

find on BIMSA website.

- No recording. (Notes upload to course webpage) it works best if you take your own note.

- You need exercises.

- Questions are most welcome.

Lecture 1

Motivation / Historical remark.

- Algebraic topology.

Hopf (1940s): H-space $(\underset{\substack{\uparrow \\ \text{Topological space}}}{X}, m)$, $\underset{\substack{\uparrow \\ \text{multiplication} \\ \uparrow \\ \text{not nec. associative}}}{m} : X \times X \to X$ continuous,

s.t. left and right multiplication are homotopic to id.

$H^\bullet(X)$ has cup product and a coproduct provided by m.

$$\Delta : H^\bullet(X) \to H^\bullet(X \times X) = H^\bullet(X) \otimes_{\Bbbk} H^\bullet(X)$$

field

If X is cpt, conn mfld, $H^\bullet(X)$ is finite-dim'l, graded, unital.

Δ and cup product are compatible.

If $\Bbbk$ is of char 0, then $H^\bullet(X)$ is an exterior algebra generated by homogeneous elements of odd degree. $H^\bullet(X) \cong H^\bullet(S^{odd} \times S^{odd} \times \cdots \times S^{odd})$

Borel, Samelson, Leray, ...

more like "bialgebra" in modern term.

- Algebraic group

Let G be a group. $R(G)$ = the space of representative functions.

$\Bbbk$-vector

$G \to \Bbbk$

Generated by matrix coefficient functions of representations of G.

$\pi : G \to GL(V)$, $\pi(g) = (u_{ij,\pi}(g))$

A natural coproduct provided by group multiplication

$$\Delta : R(G) \to R(G \times G) = R(G) \otimes_{\Bbbk} R(G)$$

s.t. $\Delta(u)(g \otimes h) = u(gh)$. Simply set

$$\Delta(u_{ij,\pi}) = \sum_{k} u_{ik,\pi} \otimes u_{kj,\pi}$$

$R(G)$ itself is an algebra w/ pointwise multiplication

+ Δ (algebra homomorphism)

+ two more algebra (anti) homomorphisms.

$S : R(G) \to R(G)$, $\varepsilon : R(G) \to \Bbbk$

$Su(g) = u(g^{-1})$ $\varepsilon u = u(1_G)$.

Another protypical example : $\mathcal{U}(\mathfrak{g})$.

- Commutative Hopf algebras are essentially the same as affine group scheme.
- If $k = \bar{k}$ and char $k = 0$, then a cocommutative Hopf algebra is of the form $\mathcal{U}(\mathfrak{g}) \# \Bbbk G$.

(A prime on Hopf algebras)

Cartier, Gabriel, Konstant, Milnor-Moore, ...

- Tensor categories

Rep(H) : good source for "nice" monoidal category.

Tannaka-Krein duality, Tannakian category, ...

- Structure theory / classification

- Quantum groups

  Yang-Baxter equation, integrable system, $U_q(\mathfrak{g})$, quasi-triangular

- Quantum topology / algebra

  Invariants of knots / links / 3-mfld, topological quantum field theory, (modular) tensor categories, ...

1. Definitions

1.1. Algebra and coalgebras

Let $\Bbbk$ be a field.

Def. A $\Bbbk$-algebra is a triple (A, μ, η), where A is a $\Bbbk$-vector space, $\mu : A \otimes_{\Bbbk} A \to A$ and $\eta : \Bbbk \to A$ are $\Bbbk$-linear maps satisfying

(a) associativity

$$\begin{array}{ccc} A \otimes_{\Bbbk} A \otimes_{\Bbbk} A & \xrightarrow{\mu \otimes_{\Bbbk} id} & A \otimes_{\Bbbk} A \\ {\scriptstyle id \otimes_{\Bbbk} \mu}\downarrow & \circlearrowright & \downarrow{\scriptstyle \mu} \\ A \otimes_{\Bbbk} A & \xrightarrow[\mu]{} & A \end{array}$$

(b) unit

$$\begin{array}{ccccc} & \nearrow^{\eta \otimes_{\Bbbk} id} & A \otimes_{\Bbbk} A & \nwarrow^{id \otimes_{\Bbbk} \eta} & \\ & & \downarrow{\scriptstyle m} & & \\ \Bbbk \otimes_{\Bbbk} A & \xrightarrow{\cong} & A & \xleftarrow{\cong} & A \otimes_{\Bbbk} \Bbbk \end{array}$$

We call μ the product (multiplication) of A, and η the unit of A.

Notation. $\otimes$ for $\otimes_{\Bbbk}$, $\eta(1) := 1_A \in A$.

__Example__. (Group algebra)

Let G be a finite group. The $\Bbbk$-vector space

$$\Bbbk G := \{ \sum_{g \in G} a_g \, g \mid a_g \in \Bbbk \}$$

has a natural algebra structure: $\mu : \Bbbk G \otimes \Bbbk G \to \Bbbk G$

$\mu(g\otimes h) = gh$ (group mult.)

$\eta(1) = 1_G$ (group unit)

$\mu(\mu\otimes id) = \mu(id\otimes\mu)$?

$\mu(\mu\otimes id)(g\otimes h\otimes k)$
$= \mu(gh\otimes k) = ghk$

$\mu(id\otimes\mu)(g\otimes h\otimes k) = \mu(g\otimes hk) = ghk$.

$\Rightarrow$ $(\Bbbk G, \mu, \eta)$ is an algebra called the group algebra of G.

<u>Example</u> (Tensor algebra)

Let V be a $\Bbbk$-vector space. Let $V^{\otimes 0} := \Bbbk$. then the vector space

$$T(V) := \bigoplus_{n\geq 0} V^{\otimes n}$$ has a natural algebra structure

induced from $V^{\otimes m}\otimes V^{\otimes n} \cong V^{\otimes(m+n)}$, $\forall\, m,n\geq 0$.

More precisely, $\mu : T(V)\otimes T(V)\to T(V)$ determined by

$$\mu\big((x_1\otimes\cdots\otimes x_m)\otimes(y_1\otimes\cdots\otimes y_n)\big) = x_1\otimes\cdots\otimes x_m\otimes y_1\otimes\cdots\otimes y_n.$$

$\forall\ x_1\otimes\cdots\otimes x_m \in V^{\otimes m}$, $y_1\otimes\cdots\otimes y_n\in V^{\otimes n}$, $\forall\ m,n\geq 0$.

$\eta(1) := 1 \in \Bbbk = V^{\otimes 0} \subseteq T(V)$.

$$T(V)\otimes T(V)\otimes T(V) \ni \underset{x_1\otimes\cdots\otimes x_{m_1}}{\underset{\shortparallel}{\vec{x}}}\otimes\vec{y}\otimes\vec{z} \ \overset{\mu(\mu\otimes id)}{\underset{\mu(id\otimes\mu)}{\rightsquigarrow}}\ \vec{x}\otimes\vec{y}\otimes\vec{z}$$

$(T(V), \mu, \eta)$ is an algebra called the **tensor algebra** of V.

DEF. For $\Bbbk$-vector spaces V, W, the **twist map** is the $\Bbbk$-linear map

$$\tau : V \otimes W \to W \otimes V$$

$\tau_{V,W}$

$$v \otimes w \mapsto w \otimes v \quad , \ \forall v \in V,\ w \in W.$$

An algebra (A, μ, η) is commutative $\Leftrightarrow$ $\mu \circ \tau_{A,A} = \mu$

DEF. A **$\Bbbk$-coalgebra** is a triple (C, Δ, ε), where C is a $\Bbbk$-vector space, $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ are $\Bbbk$-linear maps satisfying

(a) **co associativity**

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & C \otimes C \arrow[d, "\Delta \otimes id"] \\ C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

(b) **counit**

$$\begin{tikzcd} & C \arrow[dl, "1 \otimes id"'] \arrow[d, "\Delta"] \arrow[dr, "id \otimes 1"] & \\ \Bbbk \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & C \otimes \Bbbk \end{tikzcd}$$

余乘法

We call Δ the **coproduct** (comultiplication) of C, ε the **counit** of C.
We say C is **cocommutative** if $\tau \circ \Delta = \Delta$.

Abuse of notation. For any $\Bbbk$-vector space V, we sometimes automatically identify $\Bbbk \otimes V$ and $V \otimes \Bbbk$ w/ V by writing $\lambda \otimes v = \lambda v = v \otimes \lambda$

"with"

for all $v \in V$, $\lambda \in \Bbbk$.

<u>Example</u>. Let G be a finite group. Define $\Delta : \Bbbk G \to \Bbbk G \otimes \Bbbk G$ by $\Delta(g) = g \otimes g$ and $\varepsilon : \Bbbk G \to \Bbbk$, $\varepsilon(g) = 1$. for all $g \in G$.
Then $(\Bbbk G, \Delta, \varepsilon)$ is a coalgebra. 余代数

$$(\Delta \otimes id)\Delta(g) = (\Delta \otimes id)(g \otimes g) = g \otimes g \otimes g$$
$$= (id \otimes \Delta)(g \otimes g) = (id \otimes \Delta)\Delta(g)$$

$$(\varepsilon \otimes id)\Delta(g) = (\varepsilon \otimes id)(g \otimes g) = \varepsilon(g) \otimes g = g$$
$$(id \otimes \varepsilon)\Delta(g) = (id \otimes \varepsilon)(g \otimes g) = g \otimes \varepsilon(g) =$$

<u>Example</u>. Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$, and $T(\mathfrak{g})$ its tensor algebra.
The universal enveloping algebra of $\mathfrak{g}$ is defined to be
$$\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g}) / I(\mathfrak{g}),$$
where $I(\mathfrak{g})$ is the two-sided ideal of $T(\mathfrak{g})$ generated by all elements of the form $xy - yx - [x, y]$ where $x, y \in \mathfrak{g}$.
($\mathcal{U}(\mathfrak{g})$ is a $\Bbbk$-algebra)

Define $\Delta : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ by

$\Delta(x) = x \otimes 1 + 1 \otimes x$, $\forall x \in \mathfrak{g}$, extend by $\Delta(xy) = \Delta(x)\Delta(y)$.

$\Delta(1) = 1 \otimes 1$ (recall $\Bbbk = \mathfrak{g}^{\otimes 0} \subseteq T(\mathfrak{g})$)

(so that Δ is an algebra homomorphism)

$\varepsilon : \mathcal{U}(\mathfrak{g}) \longrightarrow \Bbbk$

$\varepsilon(x) = 0$, $\forall x \in \mathfrak{g}$, $\varepsilon(1) = 1$.

Then we have

$$(id \otimes \Delta)\Delta(x) = (id \otimes \Delta)(x \otimes 1 + 1 \otimes x) = x \otimes \Delta(1) + 1 \otimes \Delta(x)$$

viewed as element in $T(\mathfrak{g})$

$$= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x$$

$$= \Delta(x) \otimes 1 + \Delta(1) \otimes x = (\Delta \otimes id)\Delta(x)$$

$$(\varepsilon \otimes id)\Delta(x) = (\varepsilon \otimes id)(x \otimes 1 + 1 \otimes x) = \varepsilon(x) \otimes 1 + \varepsilon(1) \otimes x$$

$$= 1 \otimes x \; (= x)$$

viewed as element in base field

$\Rightarrow$ $(\mathcal{U}(\mathfrak{g}), \Delta, \varepsilon)$ is a coalgebra.

<u>DEF</u>. Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. A $\Bbbk$-linear map $f : C \to D$ is a **coalgebra homomorphism** if the diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varepsilon_C"'] & D \arrow[dl, "\varepsilon_D"] \\ \Bbbk & \end{tikzcd}$$

are commutative. A subspace $I \subseteq C$ is a (two-sided) **coideal** if $\Delta(I) \subseteq I \otimes C + C \otimes I$ and if $\varepsilon(I) = 0$.

<u>DEF</u> The **opposite algebra** of an algebra (A, μ, η) is the triple

$$(A^{op}, \mu^{op}, \eta^{op}) := (A, \mu \circ \tau, \eta).$$

The **coopposite coalgebra** of a coalgebra (C, Δ, ε) is the triple

$$(C^{cop}, \Delta^{cop}, \varepsilon^{cop}) := (C, \tau \circ \Delta, \varepsilon).$$