Lecture 10

Last time: $\dim(H) < \infty$, $\Bbbk = \overline{\Bbbk}$, $\operatorname{char} \Bbbk = 0$.

<u>THM 4.1</u> $S^2 = id$ if and only H is semisimple and cosemisimple.

<u>THM 4.2</u> H is semisimple if and only if H^* is semisimple.

Frobenius system : (f, r_i, l_i).

For Hopf algebras : $(\lambda^R, S^{-1}(\Lambda_2^R), \Lambda_1^R)$, $(\lambda^R, (S\Lambda_1^R)g^{-1}, \Lambda_2^R)$

Comparing Nakayama, get $S^4(h) = g(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})g^{-1}$. $\forall h \in H$.

<u>LEM 4.9</u>. A Frobenius w/ (f, r_i, l_i). Let $e \in A$ s.t. $e^2 = ce$ for some $c \in \Bbbk$, and $F \in \operatorname{End}_{\Bbbk}(eA)$. Then $c\,\mathrm{Tr}_{eA}(F) = \sum_i f(F(el_i) r_i)$.

The left regular representation on H is $H \to \operatorname{End}(H)$, $h \mapsto L_h$, $L_h(x) = hx$.
Let χ be the character of this rep, that is $\chi : H \to \Bbbk$, $\chi(h) := \mathrm{Tr}_H(L_h)$.
Clearly, $\chi \in H^*$.

<u>LEM 4.10</u>

(1) $\chi(\Theta) = \varepsilon(\Theta)$ for all $\Theta \in \int_H^L \cup \int_H^R$.

(2) $(S^*)^2(\chi) = \chi$.

(3) $\chi^2 = \dim(H)\chi$. $(\chi^2 = \chi * \chi)$ $\quad L_\Lambda(\Lambda) = \Lambda^2 = \varepsilon(\Lambda)\Lambda$

<u>Pf</u>. (1) Consider $\Lambda \in \int_H^L$. Since $L_\Lambda(h) = \alpha(h)\Lambda$, so in view of the isomorphism $H^* \otimes H \cong \operatorname{End}(H)$, L_Λ corresponds to $\alpha \otimes \Lambda$. Thus, $\chi(\Lambda) = \mathrm{Tr}_H(L_\Lambda) = \alpha(\Lambda)$ $= \varepsilon(\Lambda)$ by definition. The case for right integrals are proved similarly.

(2) $\forall h, x \in H$, $L_{S^2(h)}(x) = S^2(h) \cdot x = S^2(h \cdot S^{-2}(x))$

$= S^2 L_h S^{-2}(x)$. So $(S^*)^2(\chi)(h) = \chi(S^2(h)) = Tr_H(L_{S^2(h)}) = Tr_H(L_h) = \chi(h)$.

(3) Let $Y = H \otimes H$ as v.sp. endowed w/ the left H-action

$$k \cdot_Y (h \otimes v) := kh \otimes v \quad \forall k, h, v \in H.$$

Let $Z = H \otimes H$ as v.sp. equipped w/ two copies of left regular rep.

$$k \cdot_Z (h \otimes v) := \sum_i k_1 h \otimes k_2 v.$$

Then it is easy to check that $\varphi : Y \to Z$, $\varphi(h \otimes v) := \sum_i h_1 \otimes h_2 v$ is a left H-module isomorphism w/ inverse $\varphi^{-1}(h \otimes v) := \sum_i h_1 \otimes S(h_2) v$.

$$\begin{tikzcd}
h\otimes v \arrow[rr, mapsto] \arrow[ddd, mapsto] & & \sum_i h_1\otimes h_2 v \arrow[dd, mapsto] \\
Y \arrow[r, "\varphi"] \arrow[d, "k\cdot_Y"'] & Z \arrow[d, "k\cdot_Z"] & \\
Y \arrow[r, "\varphi"] & Z & \sum_i k_1 h_1 \otimes k_2 h_2 v \\
kh\otimes v \arrow[rr, mapsto] & & \sum_i (kh)_1 \otimes (kh)_2 v \arrow[u, equal]
\end{tikzcd}$$

$\varphi^{-1}\varphi(h \otimes v) = \varphi^{-1}(\sum_i h_1 \otimes h_2 v)$

$= \sum_i h_1 \otimes S(h_2) \cdot (h_3 \cdot v)$

$= \sum_i h_1 \otimes \varepsilon(h_2) v = h \otimes v.$

Comparing characters of Y and Z, we have $Tr_Y(k \cdot_Y) = Tr_{H\otimes H}(L_k \otimes id_H) = \chi(k) \cdot \dim(H)$
$Tr_Z(k \cdot_Z) = Tr_{H\otimes H}(\sum_i L_{k_1} \otimes L_{k_2}) = \sum_i \chi(k_1)\chi(k_2) = (\chi * \chi)(k) = \chi^2(k)$. $\forall k \in H$. ▨

<u>Prop 4.11</u> Choose $\lambda^L \in \int^L_{H^*}$ and $\Lambda^R \in \int^R_H$ s.t. $\lambda^L(\Lambda^R) = 1$. Then

(1) $Tr_H(S^2) = \varepsilon(\Lambda^R)\lambda^L(1_H)$.

(2) if $S^2 = id$, then $\dim(H) = \varepsilon(\Lambda^R)\lambda^L(1_H)$, and $\chi = \varepsilon(\Lambda^R)\lambda^L$.

Consequently, if we also have $\dim(H) \neq 0$ in $\Bbbk$, then $\lambda^L = \lambda^L(1_H)\chi/\dim(H)$.

<u>Pf</u>. Apply Lem 4.9 to H w/ $e = 1_H$ along w/ the fact that $(\lambda^L, S(\Lambda_1^R), \Lambda_2^R)$ is a Frobenius system (Prop 4.6), we have, for any $F \in End(H)$,

$$Tr_H(F) = \sum_i \lambda^L(F(\Lambda_2^R) \cdot S(\Lambda_1^R)). \quad *$$

Substitute $F = S^2$, we have

$$Tr_H(S^2) = \sum_1 \lambda^L(S^2(\Lambda_2^R)\, S(\Lambda_1^R)) = \sum_1 \lambda^L S(\Lambda_1^R \cdot S(\Lambda_2^R))$$
$$= \lambda^L S(\varepsilon(\Lambda^R)\, 1_H) = \lambda^L(1_H)\cdot \varepsilon(\Lambda^R).$$

For (2), if $S^2 = id$, then $\dim(H) = \varepsilon(\Lambda^R)\,\lambda^L(1_H)$ by (1). For the second equality, $S^2 = id$ implies

$$\begin{array}{l} \varepsilon(x)\, 1_H = S(\sum_1 x_1 S(x_2)) = \sum_1 x_2 S(x_1) \\ \quad \| \\ S(\varepsilon(x)\, 1_H) \end{array}$$

So for any $h \in H$,

apply ★ to L_h, we have $\chi(h) = \sum_1 \lambda^L(h \cdot \Lambda_2^R \cdot S(\Lambda_1^R)) = \varepsilon(\Lambda^R)\,\lambda^L(h)$.

Assume $\dim(H) \neq 0$, then $\lambda^L = \chi / \varepsilon(\Lambda^R) = \chi \cdot \dfrac{\lambda^L(1_H)}{\dim(H)}$. ▨

Rmk. Using Prop 4.7, it can be shown if $\lambda^R \in \int^R_{H^*}$, $\Lambda^R \in \int^R_H$ s.t. $\lambda^R(\Lambda^R) = 1$, then let $\Lambda^L := S^{-1}(\Lambda^R)$, one has $\lambda^R(\Lambda^L) = 1$. then $(\lambda^R, \Lambda_1^L, S(\Lambda_2^L))$ is a Frobenius system of H. Then Lem 4.9 implies

$$Tr_H(F) = \sum_1 \lambda^R(F(S(\Lambda_2^L)) \cdot \Lambda_1^L).$$

This is the formula in Radford's trace function paper.

Prop 4.13 Choose $\lambda^R \in \int^R_{H^*}$ and $\Lambda^L \in \int^L_H$ s.t. $\lambda^R(\Lambda^L) = 1$, then we have

$$Tr_{H^*}((S^*)^2) = \varepsilon(\Lambda^L)\,\lambda^R(1_H) = \dim(H)\cdot Tr_{\chi H^*}((S^*)^2).$$

Pf. The first equality is Prop 4.11 (1) applied to H^*.
For the second, apply Lem 4.9 to $e = \chi \in H^*$, (then $c = \dim(H)$ by Lem 4.10), $F = (S^*)^2|_{\chi H^*}$ and the Frobenius system used here is $(\underset{\int^L_{(H^*)^*}}{\underset{\downarrow}{\Lambda^L}}, \underset{\int^R_{(H^*)}}{\underset{\downarrow}{S(\lambda_1^R)}}, \lambda_2^R)$

$$\dim(H)\, \mathrm{Tr}_{\chi H^*}((S^*)^2) = \textstyle\sum_1 \left((S^*)^2(\chi * \lambda_2^R) * S^*(\lambda_1^R) \right) (\Lambda^L)$$

$$= \textstyle\sum_1 \left(\chi * \underbrace{(S^*)^2(\lambda_2^R) * S^*(\lambda_1^R)} \right) (\Lambda^L) = \textstyle\sum_1 \left(\chi * S^*\left(\underbrace{\lambda_1^R * S^*(\lambda_2^R)}\right)\right)(\Lambda^L)$$

$$= \varepsilon^*(\lambda^R)\, \chi(\Lambda^L) = \lambda^R(1_H) \cdot \chi(\Lambda^L) \overset{\text{LEM 4.10.}}{=} \lambda^R(1_H)\, \varepsilon(\Lambda^L) \qquad \blacksquare$$

<u>PF of THM 4.1</u>

If $S^2 = id$, then by PROP 4.11 (1), and char $\Bbbk = 0$, both $\varepsilon(\Lambda^R)$ and $\lambda^L(1_H) \neq 0$, and so both H and H^* are semisimple by Maschke's Theorem. (THM 2.1).

Conversely, if both H and H^* are semisimple, then $S^4 = id$. (by unimodularity COR 2.6 and THM 4.8). Let $\{u_j \mid j = 1, \cdots, n = \dim(H)\}$ be the eigenvalues of $(S^*)^2$ in H^*. (which can only be ± 1), and let $\{v_k \mid k = 1, \cdots, m\}$ be the eigenvalues of $(S^*)^2|_{\chi H^*}$, then by PROP 4.13, $\sum_{j=1}^n u_j = n \cdot \sum_{k=1}^m v_k$.

By semisimplicity and PROP 4.11, $\sum_{j=1}^n u_j = \mathrm{Tr}_{H^*}((S^*)^2) \neq 0$, so it is a nonzero integer multiple of n.

Since u_j can only be ± 1, $|\sum_{j=1}^n u_j| = n$. However, for at least one j, $u_j = 1$ because $(S^*)^2(\varepsilon) = \varepsilon$, which forces $u_j = 1$ for all j. Hence, $(S^*)^2 = id_{H^*}$ and $S^2 = id_H$. $\blacksquare$

Recall the following elementary fact for matrix algebras.

<u>LEM 4.14</u> Suppose $\Bbbk$ is an algebraically closed field of characteristic 0, and let $A = M_n(\Bbbk)$. Let $T \in \mathrm{Aut}_{\Bbbk}(A)$ be such that $T^m = id$ for some $m \geq 0$, then

$Tr_A(T)$ is a non-negative real number.

(Every automorphism of $M_n(\Bbbk)$ is inner).

Pf of THM 4.2. $\Bbbk = \overline{\Bbbk}$, $\operatorname{char} \Bbbk = 0$.

Assume H is semisimple, let $T = S^2$. By THM 4.8, T is of finite order. ($\dim(H) < \infty$). Now $H = \bigoplus M_{n_i}(\Bbbk) = \bigoplus A_i$, so T permutes the matrix algebras A_i. We claim that $Tr_H(T) \neq 0$.

For any i, we have the following two cases. (1) If T does not not stabilize A_i, then there exists $j \neq i$ s.t. $T(A_i) = A_j$. In this case, let $r > 0$ be the smallest integer s.t. $T^r(A_i) = A_i$, and let $B = \bigoplus_{s=0}^{r-1} T^s(A_i)$, $Tr_B(T) = 0$.

(2) If $T(A_i) = A_i$, then by LEM 4.14, $Tr_{A_i}(T) \geq 0$.

Moreover, $A_0 = \int_H \subseteq H$ is a 1-dim'l subalgebra of H, and $T(A_0) = A_0$. T is an automorphism, so $T|_{A_0} = id_{A_0}$, and $Tr_{A_0}(T) = 1$. Therefore

$$Tr_H(T) = Tr_{A_0}(T) + \underbrace{\sum_{\text{Case (2)}} Tr_{A_i}(T)}_{\geqslant 0} + \underbrace{\sum_{\text{Case (1)}} Tr_B(T)}_{=0} \geqslant 1 > 0.$$

Now we are done by PROP 4.11: $Tr_H(T) = \varepsilon(\Lambda)\lambda^R(1_H)$ for some $\Lambda \in \int_H$, $\lambda^R \in \int_{H^*}^R$, so $\lambda^R(1_H) \neq 0$, which means H^* is semisimple by Maschke. $\blacksquare$

§5. Character theory and the class equation

$\mathbb{k} = \overline{\mathbb{k}}$, $\operatorname{char} \mathbb{k} = 0$, $H =$ fin-dim'l semisimple.

Choose $\Lambda \in \int_H$ w/ $\varepsilon(\Lambda) = 1$.

Let $\operatorname{Irr}(H) = \{(V_0, \rho_0), \cdots, (V_m, \rho_m)\}$ be a complete set of irreducible left H-modules, where $V_0 = \mathbb{k}\Lambda$ is the trivial module.

Character of (V_i, ρ_i) is defined to be $\chi_i \in H^*$, $\chi_i(h) = \operatorname{Tr}_{V_i}(\rho_i(h))$.

$\chi_0 = \varepsilon$.

Let $R(H) = \operatorname{span}_{\mathbb{k}} \{\chi_i \mid 0 \le i \le m\} \subseteq H^*$.

As noted before, $V \otimes W \in {}_H\mathrm{Mod}$ if $V, W \in {}_H\mathrm{Mod}$. Semisimplicity $\Rightarrow$

$$V_i \otimes V_j \cong \bigoplus_{k=0}^{m} N_{i,j}^{k} V_k, \qquad N_{i,j}^{k} = \text{multiplicity of } V_k \text{ in } V_i \otimes V_j. \; \geq 0.$$

"fusion rules"

$$\chi_i * \chi_j = \sum_{k=0}^{m} N_{i,j}^{k} \chi_k \in H^*.$$ So $R(H)$ is a $\mathbb{k}$-algebra "character algebra".

Note that by Artin-Wedderburn and the uniqueness of trace, $f \in R(H)$ if and only if $f(kh) = f(hk)$ $\forall\, h, k \in H$. Alternatively, $f \in R(H)$ if and only if $\Delta^*(f) = \sum f_1 \otimes f_2 = \sum f_2 \otimes f_1$, i.e., f is a "cocommutative element".