Lecture 11

Last time: finished proof of Larson-Radford

§5. character theory and the class equation

H = finite dim'l semisimple Hopf algebra $/ \Bbbk$. $\quad \mathrm{char}\,\Bbbk = 0$, $\bar{\Bbbk} = \Bbbk$.

$\mathrm{Irr}(H) = \{(V_0, \rho_0), \cdots, (V_m, \rho_m)\}$ complete set of representatives isomorphism classes of irreducible left H-modules. $\quad \rho_i : H \to \mathrm{End}(V_i)$.

character: $\quad \chi_i := \mathrm{Tr}_{V_i} \circ \rho_i \in H^*$.

Example: We assume $V_0 \cong \Bbbk \Lambda$ for $\Lambda \in \int_H$ w/ $\varepsilon(\Lambda) = 1$.

H acts on $V_0 \subseteq H$ by left multiplication: $\forall h \in H$,

$$L_h(\Lambda) = \underset{\substack{\uparrow \\ \text{multiplication in } H}}{h \cdot \Lambda} = \varepsilon(h)\Lambda, \quad \text{so} \quad \chi_0(h) = \mathrm{Tr}_{V_0}(L_h) = \varepsilon(h).$$

$$\chi_0 = \varepsilon \in H^*.$$

$R(H) = \mathrm{span}_{\Bbbk}\{\chi_i \mid 0 \le i \le m\} \subseteq H^*$. As noted before, the tensor product of two H-modules is again an H-module. so for any $0 \le i, j \le m$,

$$V_i \otimes V_j \cong \bigoplus_{l=0}^{m} N_{i,j}^{l} V_l \qquad \text{"fusion rule"}$$

where $N_{i,j}^{l} = \dim \mathrm{Hom}_H(V_i \otimes V_j, V_l)$ is the multiplicity of V_l in $V_i \otimes V_j$. which are nonnegative integers.

Taking trace on both sides, we have $\quad \chi_i * \chi_j = \sum_{l=0}^{m} N_{i,j}^{l} \chi_l$.

This makes $R(H)$ a subalgebra of H^*, called the character algebra.

Note that $1_{R(H)} = 1_{H^*} = \varepsilon$. By Artin-Wedderburn and the uniqueness of trace, $f \in R(H)$ if and only if $f(hk) = f(kh)$ for all $h, k \in H$. (i.e., f is a class function).

For any left H-module V, its linear dual $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ is also a left H-module via $(\underset{\substack{\uparrow \\ V^*\text{-action}}}{h \cdot f})(v) := f(\underset{\substack{\uparrow \\ V\text{-action}}}{S(h) \cdot v})$, $\forall v \in V,\ f \in V^*$.

By assumption, H is semisimple, $S^2 = id$, so V^{**} is canonically isomorphic to V.
For any $0 \le i \le m$, $V_i^* \in \mathrm{Irr}(H)$, i.e., $\exists\ 0 \le i^* \le m$ s.t. $V_{i^*} \cong V_i^*$.
By definition, $\chi_{i^*} = \underset{\substack{\uparrow \\ \text{antipode of } H^*}}{S^*}(\chi_i) = \chi_i \circ S$ for all i.

if $H = \Bbbk G$ for a finite group G, then $\Lambda = \frac{1}{|G|} \sum_{g \in G} g$

$(\varphi | \psi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \psi(g^{-1})$

THM 5.1 (Orthogonality of characters)
Choose $\Lambda \in \int_H$ w/ $\varepsilon(\Lambda) = 1$. For any $\varphi, \psi \in H^*$, define
$$(\varphi | \psi) := (\varphi * S^*(\psi))(\Lambda) = \sum \varphi(\Lambda_1) \psi(S(\Lambda_2)).$$
then $(\chi_i | \chi_j) = \delta_{i,j}$ for all $0 \le i, j \le m$.

Sketch. By the above, $\chi_0 = \varepsilon$, so $\chi_0(\Lambda) = \underset{\substack{\| \\ 1}}{\varepsilon(\Lambda)}$. For any $0 < i \le m$, $\chi_i(\Lambda) = 0$ since $\Lambda \cdot V_i = 0$. Otherwise, there exists $v \in V_i$ s.t. $\Lambda \cdot v \ne 0$. Then $\Bbbk[\Lambda \cdot v] \subseteq V_i$ is isomorphic to V_0, which contradicts the choice of V_i.
Now $S^*(\chi_j) = \chi_{j^*}$, so
$$(\chi_i | \chi_j) = (\chi_i * \chi_{j^*})(\Lambda) = \sum_{l=0}^{m} N_{i,j^*}^{l}\, \chi_l(\Lambda) = N_{i,j^*}^{0}.$$

It is easy to check that the map $\Phi : \mathrm{Hom}_H(V_i \otimes V_{j^*}, V_0) \longrightarrow \mathrm{Hom}_H(V_i, V_j)$

determined by $\varepsilon\big(f(v\otimes w^*)\big) = w^*\big(\Phi(f)(v)\big)$ for all $v\in V_i$, $w^*\in V_{j^*}$ and $f\in \mathrm{Hom}_H(V_i\otimes V_{j^*}, V_0)$, is a linear isomorphism. By Schur's Lemma,

$$N_{i,j^*}^{0} = \dim \mathrm{Hom}_H(V_i\otimes V_{j^*}, V_0) = \dim \mathrm{Hom}_H(V_i, V_j) = \delta_{i,j}. \qquad \boxtimes$$

<u>Cor 5.2</u>. For any $0\le i,j,l\le m$, we have

- $(\cdot\,|\,\cdot)$ is symmetric and S^*-invariant
- $N_{i,j}^{l} = (\chi_l \,|\, \chi_i^{*}\chi_j)$ and
- $N_{i,j}^{l} = N_{l,j^*}^{i} = N_{j,l^*}^{i^*}$

<u>Sketch</u>. The first two statements follow immediately from THM 5.1. For the last one,

$$\underset{\Vert}{(\chi_l\, S^*(\chi_i\chi_j))(1)} = \underset{\Vert}{\underbrace{(\chi_l\, S^*(\chi_i\chi_j)\, S(\varepsilon))}(1)}$$

$$\begin{aligned}
N_{i,j}^{l} &= (\chi_l \,|\, \chi_i\chi_j) = (\chi_l\, S^*(\chi_i\chi_j) \,|\, \varepsilon) \\
&= (\chi_l\, S^*(\chi_j)\, S^*(\chi_i) \,|\, \varepsilon) = (\chi_l\, \chi_{j^*} \,|\, \chi_i) = N_{l,j^*}^{i} \\
&= (S^*(\chi_l\,\chi_{j^*}) \,|\, S^*(\chi_i)) = (\chi_j\,\chi_{l^*} \,|\, \chi_{i^*}) = N_{j,l^*}^{i^*}
\end{aligned} \qquad \boxtimes$$

Consider $R_{\mathbb{Q}}(H) := \mathrm{span}_{\mathbb{Q}}\{\chi_i \mid 0\le i\le m\}$, then S^* induces a $\mathbb{Q}$-linear involution on $R_{\mathbb{Q}}(H)$. We claim that for any $0\ne\alpha\in R_{\mathbb{Q}}(H)$, $\alpha\alpha^*\ne 0$. Indeed, write $\alpha=\sum_i a_i\chi_i$ w/ $a_i\in\mathbb{Q}$. then

$$\alpha\alpha^* = \sum_{i,j} a_i a_j\, \chi_i\chi_{j^*} = \Big(\sum_i a_i^2\Big)\varepsilon + \sum_{j>0} b_j\chi_j$$

for some $b_j\in\mathbb{Q}$. Since $\alpha\ne 0$, $\sum_i a_i^2>0$, so $\alpha\alpha^*\ne 0$. Suppose the Jacobson radical $J\subseteq R_{\mathbb{Q}}(H)$ is nonzero, then choose $0\ne\alpha\in J$, and consider $\gamma=\alpha\alpha^*\in J$. ($\gamma\ne 0$ by above), which is nilpotent. Since $\gamma^*=\gamma$, so the above computation implies $\gamma^a \neq 0$, and by induction, $\gamma^t \neq 0$ for all $t \geq 1$, contradicting the nilpotency of γ. Thus, $J = 0$, which means

LEM 5.3 $R(H)$ is a semisimple algebra. $\square$

Now we can prove the class equation for semisimple finite dim'l Hopf algebras, which is proved for Hopf C^*-algebras by G. I. Kac, and in general by Y. Zhu.

THM 5.4. Let H be a finite-dimensional semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic 0. Let $\{e_0, \cdots, e_m\}$ be a complete set of primitive orthogonal idempotents in $R(H)$, where e_0 is an integral in H^*, then for each $0 \leq i \leq m$, $\dim(e_i H^*)$ divides $\dim(H)$, and

$$\dim(H) = 1 + \sum_{i=1}^{m} \dim(e_i H^*).$$

To prove the theorem, we recall the elementary lemma.

LEM 5.5 Suppose $\mathrm{char}(\Bbbk) = 0$, $(A, (\cdot,\cdot))$ is a symmetric $((x,y) = (y,x))$ Frobenius algebra $/\Bbbk$ of dimension n w/ a dual basis $\{a_j, b_j \mid j = 1, \cdots, n\}$. Assume $A = \bigoplus_{i=1}^{N} A_i$

w/ $A_i = M_{n_i}(\Bbbk)$, and set $f_i := \sum_{j=1}^{n} Tr_i(a_j) b_j$, where Tr_i is the matrix trace in A_i. For any i, if e_i is the primitive central idempotent corresponding to A_i, then $f_i = d_i \cdot e_i$ for some $d_i \in \Bbbk^\times$.

Sketch. By assumption, for any $x, y \in A$, $(x, y) = \sum_{t=1}^{N} \gamma_t \operatorname{Tr}_t(xy)$ for some

$\gamma_t \in \Bbbk^\times$. Hence, the set $\{ \gamma_t^{-1} u^t_{r,s},\ u^t_{s,r} \mid 1 \le t \le N,\ 1 \le r, s \le n_t \}$ is a dual basis w.r.t. $(\cdot, \cdot)$, where $u^t_{r,s}$ is the usual matrix unit in A_t. Moreover, by the properties of dual bases, we have

$$\sum_{t,r,s} \gamma_t^{-1} u^t_{r,s} \otimes u^t_{s,r} = \sum_j a_j \otimes b_j .$$

Therefore,
$$f_i = \sum_{t,r,s} \operatorname{Tr}_i \left(\gamma_t^{-1} u^t_{r,s} \right) u^t_{s,r}$$
$$= \sum_r \gamma_i^{-1} \operatorname{Tr}_i (u^i_{r,r}) u^i_{r,r} = \gamma_i^{-1} \sum_r u^i_{r,r} = \gamma_i^{-1} e_i . \qquad \blacksquare$$

Now we introduce the following twist of the bilinear form $(\cdot|\cdot)$ on $R(H)$.

Choose $\Lambda \in \int_H$ s.t. $\varepsilon(\Lambda) = 1$. For $\varphi, \psi \in R(H)$, define
$$\langle \varphi, \psi \rangle := (\varphi\psi)(\Lambda) = (\varphi \mid S^*(\psi))$$

$(\varphi | \psi\gamma)$
$= (\varphi S^*(\psi) | \gamma)$
$(\varphi\psi | \gamma)$

Then $\langle , \rangle$ is nondegenerate, associative, and symmetric.

(follows from associativity of $*$)

$(x_i | x_j x_l)$ not nec. assoc.
$= (x_i x_{j^*} | x_l)$
$(x_i x_j | x_l)$

Symmetry follows from Prop 4.11 (a): since $(S^*)^2 = id_{H^*}$, then Λ is a nonzero multiple of the character of the regular rep of H^*. i.e., $(\varphi\psi)(\Lambda) = (\psi\varphi)(\Lambda)$ $\forall \varphi, \psi \in H^*$.

Moreover, it follows from THM 5.1 that $\{ S^*(x_j), x_j \}$ is a dual basis for $\langle \cdot, \cdot \rangle$. In fact, $(\langle x_0, \cdot \rangle, S^*(x_j), x_j)$ is a Frobenius system for $R(H)$. (w/ trivial Nakayama).

We present the following proof of THM 5.4 due to Lorenz.

<u>Sketch of PF of THM 5.4</u>. Let $A = R(H) = \bigoplus_{i=1}^{N} M_{n_i}(\Bbbk)$, and fix a primitive idempotent $e = e_i$ in A. We prove the divisibility first. Set

$$d := \frac{\dim(H)}{\dim(eH^*)} \in \mathbb{Q} \subset \Bbbk .$$

we claim that d is an algebraic integer. ($\exists$ monic polynomial $p(X) \in \mathbb{Z}[X]$ s.t. $p(d) = 0$).

$$\left.\begin{array}{l} \underset{\text{if } d \text{ is alg. int}}{d = \frac{p}{q}} \Rightarrow \underset{\text{monic}}{\underbrace{q\,d - p}} = 0 \end{array}\right\} \Rightarrow |q| = 1, \ d \in \mathbb{Z}.$$

Let E be the central idempotent of A which e belongs to. Then $E R(H) \cong M_{n_i}(\Bbbk) \cong \left(e R(H)\right)^{\oplus n_i}$. This implies $EH^* \cong (eH^*)^{\oplus n_i}$. Consequently,

$$\dim(EH^*) = n_i \dim(eH^*), \text{ and so}$$

$$d = \frac{n_i \dim(H)}{\dim(EH^*)}$$

Now let $f = \sum_j \mathrm{Tr}_i(S^*(x_j))\, x_j$, then by LEM 5.5, $f = \alpha E$ for some $\alpha \neq 0$ and we claim that $d = \alpha$. Indeed, let $\Lambda \in \int_H$ s.t. $\varepsilon(\Lambda) = 1$, then by PROP 4.11 (2), $\mathrm{Tr}_{H^*} = \dim(H) \cdot \Lambda$. Thus, on the one hand,

$\uparrow$ character of regular rep of H^*

$$\dim(EH^*) = \mathrm{Tr}_{H^*}(L_E) = \dim(H) \cdot E(\Lambda) = \dim(H) \cdot \alpha^{-1} \cdot f(\Lambda).$$

on the other hand, by THM 5.1,

$$f(\Lambda) = \sum_j Tr_i\,(\chi_{j^*})\,\chi_j(\Lambda) = Tr_i(\chi_0)\,\chi_0(\Lambda) = Tr_i(\varepsilon)\cdot\varepsilon(\Lambda)$$

$$= Tr_i(\varepsilon) = n_i$$

$$\hookrightarrow A = \begin{bmatrix} \square & & & \\ & \boxed{\ }^{\,Tr_i} & & \\ & & \square & \\ & & & \ddots \end{bmatrix}$$, corresponding trace when we view $\varepsilon \in A$.

i.e., $\underset{\mathbb{1}_{R(H)}}{\overset{\varepsilon}{\shortparallel}} \in R(H) \longleftrightarrow$ id matrix $\in A$.

Therefore, $$d = \frac{n_i \dim(H)}{\dim(H)\cdot\alpha^{-1}\cdot n_i} = \alpha.$$

Now we show that f is integral over $\mathbb{Z}$. i.e., there is a monic polynomial $p(X) \in \mathbb{Z}[X]$ s.t. $p(f) = 0$. Denote the elements in $\mathbb{R}$ that are integral over $\mathbb{Z}$ by $\mathcal{O}$, they form a ring called the ring of (algebraic) integers. Consider $R_{\mathbb{Z}}(H) = \mathbb{Z}[\chi_j \mid 0 \le j \le m] \subseteq R(H)$. By orthogonality, $R_{\mathbb{Z}}(H)$ is a finitely generated free $\mathbb{Z}$-module, so any element $x \in R_{\mathbb{Z}}(H)$ is integral over $\mathbb{Z}$ (see, for example, Neukirch, I. 2).

So all the eigenvalues of x are in $\mathcal{O}$, which means $Tr_i(x) \in \mathcal{O}$.

$$\underset{R_{\mathbb{Z}}(H) \subset R(H) = A = \bigoplus M_{n_j}(\mathbb{R})}{\overset{x}{\cap}}$$

Thus, all eigenvalues of $f = \sum_j Tr_i(\chi_{j^*})\,\chi_j$ are in $\mathcal{O}$. So f is integral over $\mathbb{Z}$. Let $p(X) \in \mathbb{Z}[X]$ be the monic polynomial s.t. $p(f) = 0$, then by $f = dE$ and $E^2 = E$, we have $p(d) = 0. \Rightarrow d \in \mathcal{O} \cap \mathbb{Q} = \mathbb{Z}$.

The equality in the theorem follows form $1_{R(H)} = 1_{H^*} = \varepsilon = \sum_{j=1}^{m} e_j$ $\boxtimes$