Lecture 12

Last time: Character theory of fin. dim'l s.s. Hopf algebras.

$$R(H) = \mathrm{span}_{\Bbbk}\{\chi_j\} \subseteq H^* \qquad \chi_i * \chi_j = \sum_{\ell} \underbrace{N_{i,j}^{\ell}}_{\mathbb{Z}_{\geq 0}} \chi_\ell$$

Class equation: $\Bbbk = \bar{\Bbbk}$, $\mathrm{char}(\Bbbk) = 0$. If $\{e_0, \ldots, e_m\}$ is a complete set of primitive orthogonal idempotents in $R(H)$, w/ $e_0 \in \int_{H^*}$, then for each $0 \leq i \leq m$, $\dim(e_i H^*)$ divides $\dim(H)$, and

$$\dim(H) = 1 + \sum_{i=1}^{m} \dim(e_i H^*)$$

Key ingredient: to show $d = \dfrac{\dim(H)}{\dim(e_i H^*)} \in \mathbb{Z}$, it suffices to show $d \in \mathcal{O}$ (algebraic integer). Essentially this follows from $N_{i,j}^{\ell} \in \mathbb{Z}$.

Applications / Examples.

When $H = \Bbbk G$, then the theorem reduces to the usual class equation for finite group. Indeed, we have

$$f \in R(H) \Leftrightarrow f(ghg^{-1}) = f(h),\ \forall g, h \in G \Leftrightarrow f \text{ is a class function}$$

Thus, $e_i = \sum_{g \in C_i} \delta_g$, where C_i is the i-th conjugacy class of G, and

$\dim(e_i H^*) = |C_i|$, which divides $|G|$.

When $H = (\Bbbk G)^*$, $H \cong \Bbbk^{\oplus |G|}$, and there are $|G|$ linearly independent characters. In this case, $R(H) = H^* = \Bbbk G$, and the idempotents e_i in the class equation are just the primitive idempotents of $\Bbbk G$, and $e_i H^*$ are irreducible $\Bbbk G$-modules. Then the divisibility in the class equation in this case is Frobenius' classical theorem that the degree of an irreducible G-module divides $|G|$.

THM 5.6 (Zhu). Suppose $\Bbbk$ is an algebraically closed field of characteristic 0, and H is a semisimple Hopf algebra of prime dimension p, then $H \cong \Bbbk \mathbb{Z}/p\mathbb{Z}$.

PF. By the class equation, $p = 1 + \sum_{i>0} p^{t_i}$. This forces $t_i = 0$ for all $i > 0$. Hence, there are p linearly independent characters of H, and $H \cong \Bbbk^{\oplus p}$. So, H is commutative and semisimple. so by THM 2.3.1, $H^* \cong \Bbbk \mathbb{Z}/p\mathbb{Z}$, and $H \cong \Bbbk \mathbb{Z}/p\mathbb{Z}$. ■

Actually, Zhu proved H is s.s. if $p \geq 3$, and the case $p = 2$ is trivial.

THM 5.6 holds for all $\dim(H) = p$, $\Bbbk = \bar{\Bbbk}$, $\operatorname{char}(\Bbbk) = 0$.

In 1975, Kaplansky conjectured an analog of Frobenius' theorem for Hopf algebras, which remains open. That is,

Conj. (Kaplansky). Let H be a finite dim'l semisimple Hopf algebra over an algebraically closed field of characteristic 0. Then, for any irreducible module V of H, we have $\dim(V) \mid \dim(H)$.

Chapter 3. Drinfeld double.

In this chapter, we consider arbitrary Hopf algebras over arbitrary fields, unless specified.

" Commutativity "

§1. Quasitriangular Hopf algebra.

<u>DEF</u>. An almost cocommutative Hopf algebra is a pair (H, R), where H is a Hopf algebra with bijective antipode S, and $R \in H \otimes H$ is an invertible element such that for all $h \in H$,

$$\tau \Delta(h) = R \cdot \Delta(h) \cdot R^{-1}$$

where τ is the usual swap map.

In sigma notation, $\sum_i h_2 \otimes h_1 = R \left(\sum_i h_1 \otimes h_2 \right) \cdot R^{-1}$.

<u>LEM 1.2</u> Let (H, R) be an almost cocommutative Hopf algebra, and let V, W be left H-modules. Then $V \otimes W \cong W \otimes V$ as left H-modules.

<u>PF</u>. Define $\varphi : V \otimes W \to W \otimes V$ by $v \otimes w \mapsto R^{-1}(w \otimes v)$. Then, for all $h \in H$,

$$\varphi(h \cdot (v \otimes w)) = R^{-1} \cdot \left(\sum_i h_2 \otimes h_1 \right) \cdot (w \otimes v)$$

$$= \left(\sum_i h_1 \otimes h_2 \right) \cdot \underbrace{R^{-1}(w \otimes v)}_{\varphi(v \otimes w)} = h \cdot \varphi(v \otimes w). \qquad \blacksquare$$

<u>Example</u>. Any cocommutative Hopf algebra is almost cocommutative w/ $R = 1_H \otimes 1_H$. However, such an H can also be almost cocommutative w/ a nontrivial R. On the other hand, if $H = (\mathbb{k} G)^*$ for some finite

non abelian group G, and take $g, h \in G$ s.t. $gh \neq hg$. Then by the definition of the coproduct of H, we have, for $V = \mathbb{k}\delta_g$, $W = \mathbb{k}\delta_h$, that $\delta_{gh} \cdot (V \otimes W) \neq 0$, but $\delta_{gh} \cdot (W \otimes V) = 0$, so $V \otimes W \not\cong W \otimes V$

$\hookrightarrow \delta_x = \sum_{\substack{yz = x \\ y,z \in G}} \delta_y \otimes \delta_z$.

as H-modules. Hence, H can not be almost cocommutative.

Recall from Prop 1.5.9, if H is cocommutative, then $S^2 = id$.

<u>Prop 1.4</u> Let (H, R) be almost cocommutative. Then S^2 is inner. More precisely, write $R = \sum_i a_i \otimes b_i$ and $u = \sum_i (S b_i) a_i \in H$. Then,

$u \in H^\times$, and for all $h \in H$, we have $S^2(h) = u h u^{-1} = (Su)^{-1} h (Su)$.
Consequently, $u(Su) \in Z(H)$.

$\uparrow$ center of H.

<u>Pf.</u> We first show that for any $h \in H$,

$$(*) \qquad u h = S^2(h)\, u.$$

By definition, $(R \otimes 1_H)(\sum h_1 \otimes h_2 \otimes h_3) = (\sum h_2 \otimes h_1 \otimes h_3)(R \otimes 1_H)$, and this can be rewritten as

$$\sum_i a_i h_1 \otimes b_i h_2 \otimes h_3 = \sum_i h_2 a_i \otimes h_1 b_i \otimes h_3$$

$\downarrow$

$\sum_i \sum_{(h)} a_i h_{(1)} \otimes \cdots$

$\Big\downarrow$ perm. $S^2 \otimes S \otimes id$ μ

Thus, $\sum_i S^2(h_3) \cdot S(b_i h_2) \cdot a_i h_1 = \sum_i S^2(h_3) \cdot S(h_1 b_i) \cdot h_2 a_i$

$$LHS = \sum_i S\left(b_i h_2 S(h_3)\right) \cdot a_i h_1 = \sum_i S(b_i)\, a_i\, h = u h$$

$$RHS = \sum_i S^2(h_3)\, S(b_i)\, S(h_1)\, h_2\, a_i = S^2(h)\, u.$$

and this proves $(*)$.

Next, we show that $u \in H^\times$. Write $R^{-1} = \sum_j c_j \otimes d_j$ and set

$v = \sum_j S^{-1}(d_j)\, c_j$, then, by $(*)$, we have

$$uv = \sum_j u \cdot S^{-1}(d_j)\, c_j = \sum_j S(d_j)\, u\, c_j = \sum_{i,j} S(b_i d_j)\, a_i c_j$$

$$= \mu\, \tau\, (id \otimes S)\, (\underbrace{R R^{-1}}_{\sum_{i,j} a_i c_j \otimes b_i d_j = 1_H \otimes 1_H.}) = 1_H .$$

Thus, by $(*)$, $S^2(v)\, u = 1_H$, so $u \in H^\times$, and $S^2(h) = u h u^{-1}$.
Applying S on both sides, we have $S^2(S h) = (Su)^{-1} \cdot (S h) \cdot Su$, so by the bijectivity of S, $S^2(h) = (Su)^{-1}\, h \cdot Su$. Now $S^{-2}(h) = u^{-1} h u$ $= (Su)\, h\, (Su)^{-1}$, so $u(Su) \in Z(H)$. ☒

DEF. An almost cocommutative Hopf algebra (H, R) is quasitriangular (QT) if $R = \sum_i a_i \otimes b_i$ satisfies the following conditions:

(1.1) $(\Delta \otimes id)\, R = R^{13} R^{23}$, and $(id \otimes \Delta)\, R = R^{13} R^{12}$,

where $R^{12} = \sum_i a_i \otimes b_i \otimes 1_H$, $R^{23} = \sum_i 1_H \otimes a_i \otimes b_i$, and

$R^{13} = \sum_i a_i \otimes 1_H \otimes b_i$. In this case, R is called a universal R-matrix of H. A QT Hopf algebra is triangular if $R^{-1} = \tau(R)$.

In general, for any QT Hopf algebra (H, R) w/ $R = \sum_i a_i \otimes b_i$, and for any $n \geq 2$ (given by the context), R^{jk} is understood as an n-fold tensor w/ all components 1_H except for the j and k-th position, and j, k indicates the positions of a_i and b_i

$n = 2$, $\quad R^{21} = \sum_i b_i \otimes a_i$,

$n = 4$, $\quad R^{14} \cdot R^{23} = \sum_{i,j} a_i \otimes a_j \otimes b_j \otimes b_i = R^{23} R^{14}$.

<u>Prop 1.6</u>. If (H, R) is quasitriangular, then we have

(1.2) $$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}$$

(1.3) $$(S \otimes id) R = R^{-1} = (id \otimes S^{-1}) R, \qquad (S \otimes S) R = R$$

(1.4) $$(\varepsilon \otimes id) R = 1_H \otimes 1_H = (id \otimes \varepsilon) R.$$

$$(\varepsilon \otimes id)\left(\sum_i a_i \otimes b_i\right) = \sum_i \varepsilon(a_i) \otimes b_i = \sum_i 1_H \otimes \varepsilon(a_i) b_i = 1_H \otimes \underbrace{\sum_i \varepsilon(a_i) b_i}$$
$$= 1_H \otimes \underbrace{1_H}$$

<u>Pf</u>. First, by definition, we have

$$R^{12} \underbrace{R^{13} R^{23}} = R^{12} \cdot (\Delta \otimes id)(R) = \sum_i R \cdot \Delta(a_i) \otimes b_i$$

$$= \sum_i \tau\Delta(a_i) \cdot R \otimes b_i = (\tau \otimes id) \underbrace{(\Delta \otimes id)(R)} \cdot R^{12}$$

$$= (\tau \otimes id)(R^{13} R^{23}) \cdot R^{12} = R^{23} R^{13} R^{12}$$

$\sum_i a_i \otimes 1 \otimes b_i \;\to\; 1 \otimes a_i \otimes b_i \;\to\; R^{23}$

$1 \otimes a_j \otimes b_j \;\to\; a_j \otimes 1 \otimes b_j \;\to\; R^{13}$

$$\sum \varepsilon(a_{i,1}) \otimes a_{i,2} \otimes b_i = \sum 1_H \otimes a_i \otimes b_i$$

To prove (1.4), note that $1_H \otimes R = (\varepsilon \otimes id \otimes id)(\Delta \otimes id) R$

$$= (\varepsilon \otimes id \otimes id)(R^{13} R^{23}) = \sum_{i,j} \varepsilon(a_i) \otimes a_j \otimes b_i b_j$$

$$\sum_i 1_H \otimes \overset{1_H}{\varepsilon(a_i)} \otimes b_i \cdot \sum_j 1 \otimes a_j \otimes b_j$$

$= (1_H \otimes (\varepsilon \otimes id) R)(1_H \otimes R)$, cancelling $1_H \otimes R$ from both sides, we have $1_H \otimes 1_H \otimes 1_H = 1_H \otimes (\varepsilon \otimes id) R$. Applying μ on both sides, we have $1_H \otimes 1_H = (\varepsilon \otimes id) R$. The other equality can be similarly proved.

Finally, we have

$$R \cdot (S \otimes id)(R) = \sum_{i,j} a_i S(a_j) \otimes b_i b_j$$

$$= (\mu \otimes id)(id \otimes S \otimes id)(R^{13} R^{23}) = (\mu \otimes id)(id \otimes S \otimes id)(\Delta \otimes id)(R)$$

$$= (\varepsilon \otimes id)(R) = 1_H \otimes 1_H \Rightarrow (S \otimes id)(R) = R^{-1}$$

Now consider H^{cop} w/ coproduct $\Delta^{cop} = \tau \Delta$ and antipode $S^{cop} = S^{-1}$.

Since τ is an algebra homomorphism on $H \otimes H$,

$$\tau(R)\, \Delta^{cop}(h) = \tau(R \cdot \Delta(h)) = \tau(\tau\Delta(h) \cdot R) = \Delta(h) \cdot \tau(R).$$

for all $h \in H$. Moreover, it is easy to check that $(H^{cop}, \tau(R))$ is QT. Therefore, by the above, we have $(S^{cop} \otimes id)(\tau(R)) = (S^{-1} \otimes id)(\tau(R)) = \tau(R)^{-1}$. applying τ, we have $(id \otimes S^{-1})(R) = R^{-1}$. Finally,

$$\begin{aligned}(S \otimes S)(R) &= (id \otimes S)(S \otimes id)(R) \left(= (id \otimes S)(R^{-1})\right) \\ &= (id \otimes S)(id \otimes S^{-1})(R) = R.\end{aligned}$$

$\blacksquare$