Lecture 13

Last time : QT Hopf algebra. (H, R)

$$\left.\begin{array}{ll}(QT1) & R\in H\otimes H \text{ invertible}\\ (QT2) & \tau\Delta(h)=R\cdot\Delta(h)\cdot R^{-1},\quad \forall h\in H\end{array}\right\}\text{ Almost cocommutative}$$

$$(QT3)\qquad (\Delta\otimes id)R = R^{13}R^{23}, \qquad (id\otimes\Delta)R = R^{13}R^{12}$$

$$R=\sum_i a_i\otimes b_i, \qquad R^{jk}=\sum_i 1_H\otimes\cdots\otimes 1_H\otimes \underset{\substack{\uparrow\\ j\text{th pos.}}}{a_i}\otimes 1_H\otimes\cdots\otimes \underset{\substack{\uparrow\\ k\text{th pos.}}}{b_i}\otimes 1_H\cdots$$

Lem 1.2' If (H, R) is almost cocommutative, then $V\otimes W\cong W\otimes V$ for all left H-modules V and W.

Sketch. Check
$$\varphi: V\otimes W\to W\otimes V \quad \text{is an } H\text{-module isom.}$$
$$v\otimes w\mapsto \tau(R\cdot(v\otimes w))$$

Prop 1.6. If (H, R) is QT, then

$$(QYBE)\qquad R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$$

$$(S\otimes id)R = R^{-1} = (id\otimes S^{-1})R, \qquad (S\otimes S)R = R$$

$$(\varepsilon\otimes id)R = 1_H = (id\otimes\varepsilon)R \qquad \Big((\varepsilon\otimes id)R=\sum_i \varepsilon(a_i)\, b_i \quad \text{if } R=\sum_i a_i\otimes b_i\Big)$$

identify $\gamma\in k$ w/ $\eta(\gamma)=\gamma\cdot 1_H\in H$.

$$\text{''}\ \varepsilon(a_i)\otimes b_i \to \varepsilon(a_i)1_H\otimes b_i\ \text{''}$$
$$\downarrow$$
$$\varepsilon(a_i)b_i$$

QYBE : Quantum Yang-Baxter equation.

The quantum universal enveloping algebras $U_q(\mathfrak{g})$ were introduced to study matrix solutions to QYBE.

Note that, R acts on $H \otimes H$ by left multiplication L_R. Write

$$B^{ij} := \tau^{ij} \circ L_{R^{ij}} \quad \in \mathrm{Aut}(H^{\otimes 3})$$

τ^{ij} : interchanges tensor factors in the obvious way.

In general, given $n \in \mathbb{Z}_{\geq 2}$, the automorphism on $H^{\otimes n}$ induced by some $\sigma \in S_n$ (in cycle form) is denoted by τ^{σ}. For example, $\tau^{12}\tau^{23}\tau^{12} = \tau^{13}$

Then, for any $s, t, u \in H$, QYBE implies

$$B^{12}B^{23}B^{12}(s \otimes t \otimes u) = B^{12}B^{23}\left(\tau^{12}(R^{12}) \cdot \tau^{12}(s \otimes t \otimes u)\right)$$

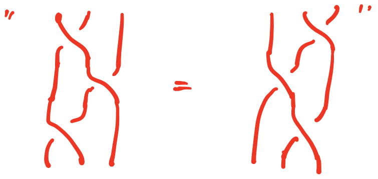
$$= \tau^{12}(R^{12}) \cdot \tau^{12}\tau^{23}(R^{23}) \cdot \tau^{12}\tau^{23}\tau^{12}(R^{12}) \cdot \tau^{12}\tau^{23}\tau^{12}(s \otimes t \otimes u)$$

$$= \tau^{13}(R^{23}R^{13}R^{12}) \cdot \tau^{13}(s \otimes t \otimes u)$$

$$= \tau^{13}(R^{12}R^{13}R^{23}) \cdot \tau^{13}(s \otimes t \otimes u) = B^{23}B^{12}B^{23}(s \otimes t \otimes u).$$

Let $\beta = \tau \circ L_R \in \mathrm{Aut}(H \otimes H)$, then the above implies

(Braid relation) $\quad (\beta \otimes id)(id \otimes \beta)(\beta \otimes id) = (id \otimes \beta)(\beta \otimes id)(id \otimes \beta)$



In general, for any vector space V over $\Bbbk$, a linear automorphism $\beta \in \mathrm{Aut}(V \otimes V)$ satisfying the braid relation in $\mathrm{Aut}(V^{\otimes 3})$ is called an R-matrix for V. Finding R-matrices for arbitrary vector spaces can be computationally challenging. as the braid relation is a system of degree 3 homogeneous equations in a huge number of variables. However, if a vector space has an additional structure of an H-module for some QT Hopf algebra (H, R), then it automatically has an R-matrix afforded by R.

Indeed, for any left H-module V, define $\beta_{V,V} : V \otimes V \longrightarrow V \otimes V$

$$v \otimes v' \mapsto \tau(R \cdot (v \otimes v'))$$

then the above implies that $\beta_{V,V}$ is an R-matrix for V. Moreover, LEM 1.2' implies that $\beta_{V,V} \in \mathrm{Aut}_H(V \otimes V)$.

Another special property of QT Hopf algebra is given as follows.

THM 1.7 Let (H,R) be QT. Write $R = \sum_i a_i \otimes b_i$, and $u = \sum_i (Sb_i) a_i$. Then for all $h \in H$, $S^4(h) = y h y^{-1}$, where $y = u(Su)^{-1} \in G(H)$.

part of statement.

PF. The equation involving S^4 follows immediately from PROP 1.4

$S^2(h) = uhu^{-1} = (Su)^{-1} h (Su)$

so it remains to show $y \in \overset{\text{grouplike}}{\overset{\downarrow}{G(H)}}$, and it suffices to show $\Delta(y) = y \otimes y$. (as $\varepsilon(y) = \varepsilon(u(Su)^{-1}) = 1$. by def and PROP 1.4).

Firstly, we show that

$$(**) \qquad \Delta(u) = (R^{21}R)^{-1}(u \otimes u) = (u \otimes u)(R^{21}R)^{-1}.$$

By def, for any $h \in H$,

$$R^{21}R(\Delta(h)) = \tau(R) \cdot \tau\Delta(h) \cdot R = \tau(R \cdot \Delta(h))R = \tau(\tau\Delta(h) \cdot R)R$$

$$= \Delta(h) \cdot R^{21}R$$

so it suffices to show $\Delta(u) R^{21} R = u \otimes u$. Again by def,

$$\Delta(u) = \sum_i \Delta(Sb_i) \cdot \Delta(a_i) = \sum_i (S \otimes S)(\tau\Delta(b_i)) \cdot \Delta(a_i)$$

so $$\Delta(u) R^{21} R = \sum_i (S \otimes S)(\tau\Delta(b_i)) \cdot R^{21} R \cdot \Delta(a_i)$$

Now we make $H^{\otimes 2}$ into a right $H^{\otimes 4}$-module by defining

$$(h \otimes k) \diamond (c \otimes d \otimes e \otimes f) := (Se)hc \otimes (Sf)kd.$$ (check it's indeed an action)

$$\sum_i (S \otimes S)(b_{i,(2)} \otimes b_{i,(1)})(a_{i,(1)} \otimes a_{i,(2)})$$
$$\uparrow$$

Then $$\Delta(u) R^{21} R = \sum_i S(b_{i,(2)}) b_k a_j a_{i,1} \otimes S(b_{i,(1)}) a_k b_j a_{i,(2)}$$

$$\downarrow \quad \hookrightarrow \ldots$$
$$\sum_k b_k \otimes a_k$$

$$= \underbrace{R^{21}}_{\in H^{\otimes 2}} \diamond [\underbrace{R^{12} \cdot (\Delta \otimes \tau\Delta)(R)}_{\in H^{\otimes 4}}]$$

Recall by (QT3), $(id \otimes \tau\Delta) R = \overset{\substack{a,1,b \\ \downarrow \times \\ a\ b\ 1}}{(id \otimes \tau)} (R^{13} R^{12}) = R^{12} R^{13}$, and so

$$(\Delta \otimes \tau\Delta) R = (id \otimes id \otimes \tau\Delta)(\Delta \otimes id) R$$

$$= (id \otimes id \otimes \tau\Delta)(R^{13}) \cdot (id \otimes id \otimes \tau\Delta)(R^{23})$$

$$= \tau^{12}\left(\tau^{12}\left(\sum_i a_i \otimes 1_H \otimes \tau\Delta(b_i)\right)\right) \cdot (id \otimes id \otimes \tau\Delta)(R^{23})$$

$$= \tau^{12}\left(\sum_i 1_H \otimes a_i \otimes \tau\Delta(b_i)\right) \cdot (id \otimes id \otimes \tau\Delta)(R^{23})$$

$$= \tau^{12}\left(1_H \otimes (id \otimes \tau\Delta)(R)\right) \cdot (id \otimes id \otimes \tau\Delta)(R^{23})$$

$$= \underbrace{\tau^{12}\left(1_H \otimes (R^{12} R^{13})\right)}_{\downarrow} \cdot \underbrace{(id \otimes id \otimes \tau\Delta)(R^{23})}$$

$$= \underbrace{R^{13}}\, R^{14}\, \boxed{R^{23}\ R^{24}}$$

$$= R^{13} R^{23} R^{14} R^{24}$$

"a 1 1 b"
"1 a b 1"

So by QYBE, $R^{12} \cdot (\Delta \otimes \tau\Delta) R = R^{23} R^{13} R^{12} R^{14} R^{24}$. Also, by Prop 1.6

$R^{-1} = (id \otimes S^{-1}) R = \sum_i a_i \otimes S^{-1} b_i$, so

$$\underbrace{R^{21}}_{\in H^{\otimes 2}} \diamond \underbrace{R^{23}}_{\in H^{\otimes 4}} = \sum_i (b_i \otimes a_i) \diamond (1_H \otimes a_j \otimes b_j \otimes 1_H)$$

$$= \sum_i S(b_j) b_i \otimes a_i a_j = (S \otimes id)\left[\left(\sum_i S^{-1} b_i \otimes a_i\right) \cdot \left(\sum_i b_j \otimes a_j\right)\right]$$

$$= (S \otimes id)\left[\tau(R^{-1}) \cdot \tau(R)\right] = (S \otimes id)(1_H \otimes 1_H) = 1_H \otimes 1_H$$

So $R^{21} \diamond (R^{23} R^{13}) = (1_H \otimes 1_H) \diamond R^{13} = u \otimes 1_H$. Thus,

$$R^{21} \diamond (R^{23} R^{13} R^{12} R^{14}) = (u \otimes 1_H) \diamond (R^{12} R^{14}) = \sum u a_i a_j \otimes S(b_j) b_i$$

$$= (u \otimes 1_H) \cdot \underbrace{\sum_i a_i a_j \otimes S(b_j) b_i}_{= 1_H \otimes 1_H} = u \otimes 1_H$$

Finally, $(u \otimes 1_H) \diamond R^{24} = u \otimes u$, proving $(**)$.

Now we have

$$\Delta(Su) = (S \otimes S)(\tau \Delta(u)) = (S \otimes S)\left[\tau(R^{21} R)^{-1} \cdot (u \otimes u)\right] \quad \text{by } (**).$$

$$= (S \otimes S)(u \otimes u) \cdot (S \otimes S)(R R^{21})^{-1} = (Su \otimes Su) \cdot (R^{21} R)^{-1} \quad \text{by } \underline{\text{Prop 1.6}}.$$

Therefore, $\Delta(y) = \Delta(u) \Delta(Su)^{-1}$

$$= (u \otimes u)(R^{21} R)^{-1} \cdot (R^{21} R) \cdot \left((Su)^{-1} \otimes (Su)^{-1}\right)$$

$$= y \otimes y$$

▨

Rmk. If (H, R) is QT w/ $\dim(H) < \infty$, let $g \in H$, $\alpha \in H^*$ be the distinguished group like elements, $y = u(Su)^{-1}$ as above. If $\tilde{\alpha} := (\alpha \otimes id) R \in H$, then $gy = yg = \tilde{\alpha}$. The proof can be found in [Radford, On the antipode of QTHA].

DEF. If (H, R) and (H', R') are QT, then they are isomorphic as QT Hopf algebra if and only if there exists a Hopf algebra isom $f : H \to H'$ s.t. $R' = (f \otimes f)(R)$. Two universal R-matrices R, R' on H are equivalent if $(H, R) \cong (H, R')$ as QT Hopf algebras.

Example. For any cocommutative Hopf algebra H, $(H, 1_H \otimes 1_H)$ is a QT Hopf algebra. but H may have other universal R-matrices. For example, $H = \Bbbk \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{char}(\Bbbk) \neq 2$, then a nontrivial R is given by

$$R = \frac{1}{2}\left(e \otimes e + e \otimes g + g \otimes e - g \otimes g \right)$$

where we write $\mathbb{Z}/2\mathbb{Z} = \{e, g\}$ multiplicatively.

Example. Let $H = T_4(-1)$ and assume $\mathrm{char}(\Bbbk) = 2$. Then H has a one-parameter family of universal R-matrices : for $\theta \in \Bbbk$, define

$$R_\theta := \frac{1}{2}\left(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g \right) + \frac{\theta}{2}\left(x \otimes x - x \otimes gx + gx \otimes x + gx \otimes gx \right)$$

Radford showed they are indeed universal R-matrices. and if $\theta \neq 0$, then $(H, R_\theta) \not\cong (H, R_0)$.

"**Example**". Let $q \in \mathbb{C}^\times$ be a nonzero complex number that is not a root of unity. The Lie algebra $\mathfrak{sl}_2$ of 2×2-traceless matrices has a basis $\{e, f, h\}$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is easy to see that $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$.

Recall $U(sl_2)$ is a Hopf algebra w/

$$\Delta(x)=x\otimes 1+1\otimes x, \quad S(x)=-x, \quad \varepsilon(x)=0 \quad \text{for all } x\in sl_2.$$

The quantum enveloping algebra $\mathcal{U}_q(sl_2)$ is defined as follows.

As an algebra, $\mathcal{U}_q(sl_2)=\langle E,F,K,K^{-1}\rangle$ subject to the following relations

$$KK^{-1}=K^{-1}K=1, \quad KE=q^2EK, \quad KF=q^{-2}FK, \quad EF-FE=\frac{K^2-K^{-2}}{q^2-q^{-2}}.$$

A Hopf algebra structure on $\mathcal{U}_q(sl_2)$ is given by

$$\Delta(E)=E\otimes K^{-1}+K\otimes E, \quad S(E)=-q^{-2}E, \quad \varepsilon(E)=0$$

$$\Delta(F)=F\otimes K^{-1}+K\otimes F, \quad S(F)=-q^{2}F, \quad \varepsilon(F)=0$$

$$\Delta(K)=K\otimes K, \quad S(K)=K^{-1}, \quad \varepsilon(K)=1$$

(For details, see standard textbooks on quantum groups. e.g. Kassel)

To understand the relation between $U(sl_2)$ and $\mathcal{U}_q(sl_2)$, one can work w/ "topological algebras" over $\mathbb{C}[[\hbar]]$, the ring of power series in the variable $\hbar$. Then one can think of $q=\exp(\hbar)$ and $K=\exp(\hbar h)$. In general, one can define $\mathcal{U}_q(\mathfrak{g})$ for complex simple Lie algebra $\mathfrak{g}$, and they are the motivating examples of QT Hopf algebras. However, they are not QT in the strict sense of our definition. One should again work w/ $\mathbb{C}[[\hbar]]$ and find a universal R-matrix in the "topological tensor product of the topological algebra $\mathcal{U}_q(\mathfrak{g})$ w/ itself". Nevertheless, $\forall$ fin. dim'l rep $\rho:\mathcal{U}_q(\mathfrak{g})\to \mathrm{End}(V)$, then

$R_\rho := (\rho\otimes\rho)R \in M_n(\mathbb{C})\otimes M_n(\mathbb{C})$ is a matrix solution to QYBE.