Lecture 14

Last time : QT Hopf algebra , QYBE

§2. Coquasitriangular Hopf algebras

DEF. A coquasitriangular (CQT) Hopf algebra is a pair (H, r), where H is a Hopf algebra and $r \in (H \otimes H)^*$ is a bilinear form on H, satisfying the following conditions :

- (CQT 1) r is invertible in $(H \otimes H)^*$.
- (CQT 2) $r * \underbrace{\mu} = \mu\tau * r$, where we identify r w/ $\eta \circ r \in \mathrm{Hom}_k(H\otimes H, H)$

  in $\mathrm{Hom}_k(H\otimes H, H)$

  multiplication . $\mu : H \otimes H \to H$ unit
- (CQT 3) $r \circ (\mu \otimes id) = r^{13} * r^{23}$, $r \circ (id \otimes \mu) = r^{13} * r^{12}$

where $r^{12} = r \otimes \varepsilon$, $r^{23} = \varepsilon \otimes r$, and $r^{13} = (\varepsilon \otimes r)(\tau \otimes id)$.

The pair is called almost commutative if it only satisfies (CQT 1) and (CQT 2).

" when H is fin-dim'l,
$r = \sum_i \varphi_i \otimes \psi_i \in H^* \otimes H^*$
$r^{13} = \sum_i \varphi_i \otimes 1_{H^*} \otimes \psi_i$
$r^{13}\underline{(a \otimes b \otimes c)}$
$= \sum \varphi_i(a)\, \varepsilon(b)\, \psi_i(c)$
$= \sum_i \varepsilon(b)\, \varphi_i(a)\, \psi_i(c)$ "
$= (\varepsilon \otimes r)\,(b \otimes a \otimes c)$

Write above in terms of H.

Let $a, b, c, d \in H$ be arbitrary elements. then $r^{13}(a \otimes b \otimes c) = \varepsilon(b)\, r(a \otimes c)$

(CQT 1) says there exists $r' \in (H \otimes H)^*$ s.t.

$$(r * r')(a \otimes b) = \sum_i r(a_1 \otimes b_1) \cdot r'(a_2 \otimes b_2) = \sum_i r'(a_1 \otimes b_1)\, r(a_2 \otimes b_2)$$

$$= (r' * r)(a \otimes b) = \varepsilon(a)\, \varepsilon(b).$$

(CQT 2) requires $\sum r(a_1 \otimes b_1) \cdot a_2 b_2 = \sum r(a_2 \otimes b_2)\, b_1 a_1$

$$\underset{(r * \mu)(a \otimes b)}{\underbrace{\sum r(a_1 \otimes b_1) \cdot a_2 b_2}} = \underset{(\mu\tau * r)}{\underbrace{\sum r(a_2 \otimes b_2)\, b_1 a_1}}$$

(CQT 3) says $(r \circ (\mu \otimes id))(a \otimes b \otimes c) = (r^{13} * r^{23})(a \otimes b \otimes c)$

$$r(ab \otimes c) = \sum r(a \otimes c_1)\, r(b \otimes c_2)$$

Note: the notation here is different from Montgomery's book.

Let (H, r) be CQT. If H is finite-dim'l, then $(H \otimes H)^* = H^* \otimes H^*$ as Hopf algebras. In this case, (CQT 1) and (CQT 2) imply that (H^*, r) is almost cocommutative. In fact, (CQT 2) is equivalent to

$r * \Delta^*(f) = \tau \Delta^*(f) * r$ for all $f \in H^*$.

$(r * \Delta^*(f))(a \otimes b) = \sum r(a_1 \otimes b_1)\, \Delta^*(f)(a_2 \otimes b_2) = \sum r(a_1 \otimes b_1)\, f(a_2 b_2)$

$= \cdots = (\tau \Delta^*(f) * r)(a \otimes b)$.

Moreover, (CQT 3) is exactly (QT 3) for H^*.

<u>Prop 2.2</u>. Let H be a finite-dim'l Hopf algebra. Then a pair (H, r) is CQT if and only if (H^*, r) is QT. ▨

<u>Example</u>. Let H be any commutative Hopf algebra, then $(H, \varepsilon \otimes \varepsilon)$ is CQT.

<u>Example</u>. Let $H = \Bbbk G$ be a group algebra, then H admits a CQT structure $r \in (H \otimes H)^*$ if and only if for all $g, h, k \in G$,

- $r(g \otimes h) \in \Bbbk^{\times}$
- $gh = hg$
- $r(gh \otimes k) = r(g \otimes k)\, r(h \otimes k)$, $\quad r(g \otimes hk) = r(g \otimes h)\, r(g \otimes k)$.

Therefore, in this case, (H, r) is CQT if and only if the following conditions hold: (1) G is abelian; (2) r is induced from a bicharacter of G.

$$\left(\begin{array}{l} \chi : G \times G \to \Bbbk^{\times} \\ \chi(gh, k) = \chi(g, k)\, \chi(h, k) \\ \chi(g, hk) = \chi(g, h)\, \chi(g, k) \end{array} \right)$$

By definition, bicharacters of G naturally corresponds to group homomorphisms from G to $\hat{G} := \mathrm{Hom}(G, \Bbbk^{\times})$.

Now suppose G is a finite abelian group of order n, and suppose $\Bbbk$ contains a primitive n-th root of unity. Then $G \cong \hat{G}$ and $(\Bbbk G)^{*} = \Bbbk \hat{G}$. By Prop 2.2, the QT stuctures on $\Bbbk G$ are in one-to-one correspondence the bicharacters on $\hat{G}$.

For example, consider $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$ written multiplicatively. Write $\hat{G} = \{\varepsilon, \gamma\}$, where ε is the trivial character, and $\gamma(g) = -1$. Then there is only one nontrivial bicharacter z on $\hat{G}$, namely, $z(\gamma, \gamma) = -1$, and all other z-values are 1. Suppose R_z is the QT structure on $\Bbbk G$ corresponding to z, and write

$$R_z = \alpha_1 (e \otimes e) + \alpha_2 (e \otimes g) + \alpha_3 (g \otimes e) + \alpha_4 (g \otimes g) \in \Bbbk G \otimes \Bbbk G.$$

By the above discussions, we have

$$(\varepsilon \otimes \varepsilon)(R_z) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = z(\varepsilon, \varepsilon) = 1$$

$$(\varepsilon \otimes \gamma)(R_z) = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = z(\varepsilon, \gamma) = 1$$

$$(\gamma \otimes \varepsilon)(R_z) = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = z(\gamma, \varepsilon) = 1$$

$$(\gamma \otimes \gamma)(R_z) = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = z(\gamma, \gamma) = -1$$

The above system of linear equations has a unique solution, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}$, $\alpha_4 = -\frac{1}{2}$, and R_z is exactly the R in the previous section.

LEM 2.5 Let (H, r) be an almost commutative Hopf algebra, and let V, W be left H-comodules. Then $V \otimes W \cong W \otimes V$ as left comodules.

Pf. Define $\beta_{V,W} : V \otimes W \longrightarrow W \otimes V$ by $v \otimes w \mapsto \sum_i r(\underbrace{w_{-1} \otimes v_{-1}}_{\in H})\, \underbrace{w_0}_{\in W} \otimes \underbrace{v_0}_{\in V}$

Since r is invertible, $\beta_{V,W}$ is a linear isom. It remains to show that it is an H-comodule map. Let ρ be the coaction of H.

$$\begin{tikzcd}
V \otimes W \arrow[r, "\beta_{V,W}"] \arrow[d, "\rho_{V\otimes W}"'] \arrow[dr, phantom, "?"] & W \otimes V \arrow[d, "\rho_{W \otimes V}"] \arrow[r, "\rho_W \otimes \rho_V"] \arrow[d, bend left=60] & H \otimes W \otimes H \otimes V \arrow[d, "id \otimes \tau \otimes id"] \\
H \otimes (V \otimes W) \arrow[r, "id_H \otimes \beta_{V,W}"] & H \otimes (W \otimes V) & H \otimes H \otimes W \otimes V \arrow[l, "\mu \otimes id \otimes id"']
\end{tikzcd}$$

$$(\rho_{W\otimes V} \circ \beta_{V,W})(v \otimes w) = \rho_{W \otimes V}\left(\sum_i r(w_{-1} \otimes v_{-1})\, w_0 \otimes v_0\right)$$

$$= \sum_i r(w_{-2} \otimes v_{-2})\, \underbrace{w_{-1}\, v_{-1}}_{\in H} \otimes \underbrace{w_0}_{\in W} \otimes \underbrace{v_0}_{\in V}$$

$$\overset{(CQT\ 2)}{=} \cdots = ((id \otimes \beta_{V,W}) \circ \rho_{V \otimes W})(v \otimes w)$$ ▨

Set $V = W$ in the above lemma, then we can show that $\beta_{V,V}$ is an R-matrix for V. In fact, it is clear that one can define a CQT bialgebra in the same way as we did for Hopf algebras. Then LEM 2.5 holds also for

CQT bialgebras, i.e., a CQT bialgebra gives rise to an R-matrix on every comodule.

The famous FRT construction, due to Faddeev, Reshetikhin and Takhtadjian, says that the converse is true. Namely, if any $\overset{\text{fin.-dim'l}}{\vee}$ $\Bbbk$-vector space V admits an R-matrix $\beta \in \mathrm{Aut}(V \otimes V)$, then $\exists$ a CQT bialgebra (A_β, r) s.t. V is a left A_β-comodule, and $\beta \in \mathrm{Aut}_{A_\beta}(V \otimes V)$, and

$\beta = \beta_{V,V}$ defined in LEM 2.5. The construction is briefly described as follows.

Let $\{ v_i \mid 1 \le i \le N \}$ be a basis for V, and write the matrix presentation of β w/ r.t. $\{ v_i \otimes v_j \mid 1 \le i, j \le N \}$ for $V \otimes V$ by

$$\beta(v_i \otimes v_j) = \sum_{k,l} b_{ij}^{kl} \; v_k \otimes v_l$$

Consider the free algebra $F_\beta = \Bbbk \langle X_s^t \mid 1 \le s, t \le N \rangle$ generated by a family of indeterminines X_s^t, and let I_β be the two-sided ideal of F_β generated by all elements of the form

$$\left(\sum_{1 \le k, l \le N} b_{ij}^{kl} \, X_k^s \, X_l^t \right) - \left(\sum_{1 \le m, n \le N} X_i^m \, X_j^n \, b_{mn}^{st} \right)$$

where i, j, s, t run over the index set $\{1, \cdots, N\}$. Then the quotient

$A_\beta := F_\beta / I_\beta$ has an (essentially unique) CQT algebra structure determined by $\Delta(X_s^t) = \sum_{i=1}^{N} X_s^i \otimes X_i^t$, $\varepsilon(X_s^t) = \delta_{s,t}$.

and $r \in (A_\beta \otimes A_\beta)^*$ is determined $r(X_i^k \otimes X_j^l) = b_{ji}^{kl}$.

For example, suppose $\Bbbk = \mathbb{C}$, and $\dim(V) = 2$ w/ basis $\{v_1, v_2\}$. It is easy to check, w/ r.t. $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$, the matrix

$$\beta = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \text{where } q \in \mathbb{C}^\times \text{ is not a root of unity,}$$

is an R-matrix for V. Denote $X_1^1 := a$, $X_2^1 := b$, $X_1^2 := c$, $X_2^2 := d$. then the algebra A_β is generated by a, b, c, d, subject to the relation

$$\beta \cdot \begin{pmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{pmatrix} = \begin{pmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{pmatrix} \cdot \beta$$

It is easy to see that

$$A_\beta = \mathbb{C}\langle a, b, c, d \mid ba = qab,\ ca = qac,\ cb = bc,\ db = qbd,\ da - ad = (q-q^{-1})bc \rangle$$

w/ the bialgebra structure defined above. As a bialgebra, this is called the coordinate ring of quantum 2×2-matrices, and is denoted by $\mathcal{O}_q(M_2(\mathbb{C}))$

in Mont. book. (compare w/ $\mathcal{O}(M_2(\mathbb{C}))$ in our Section 1.4)

One can introduce a notion of q-determinant so as to define $\mathcal{O}_q(SL_2(\mathbb{C}))$ and $\mathcal{O}_q(GL_2(\mathbb{C}))$. In fact, it can be shown (e.g. Kassel's book) $\mathcal{O}_q(SL_2(\mathbb{C}))$ is "dual" to $\mathcal{U}_q(\mathfrak{sl}_2)$ in an appropriate sense.

§3. Drinfeld double.

DEF. Let H be any Hopf algebra w/ bijective antipode S w/ inverse $\bar{S}$, and let $h \in H$, $f \in H^*$ be arbitrary elements. The left coadjoint action of H on H^* is given by

$$h \rightharpoondown f := \sum h_1 \rightharpoonup f \leftharpoonup \bar{S}(h_2)$$

The right coadjoint action of H on H^* is given by

$$f \leftharpoondown h := \sum \bar{S}(h_1) \rightharpoonup f \leftharpoonup h_2.$$

Rmk. The coadjoint actions are related to the obvious adjoint actions of H on itself. $\forall\, h, k \in H$, $ad^L(h)(k) := \sum h_1 k S(h_2)$ and $ad^R(h)(k) = \sum S(h_1) k h_2$, then it is easy to check

$$\langle h \rightharpoondown f, k \rangle = \langle f, ad^L(\bar{S}(h))(k) \rangle \qquad \text{and}$$

$$\langle f \leftharpoondown h, k \rangle = \langle f, ad^R(\bar{S}(h))(k) \rangle$$

For example, if $H = \mathbb{k}G$ for a finite group G, then $H^* = (\mathbb{k}G)^*$. In this case, for any $x, y, z \in G$, $(y \rightharpoondown \delta_x)(z) = \delta_{x, zy} = \delta_{xy^{-1}}(z)$

so $y \rightharpoondown \delta_x = \delta_{xy^{-1}}$. Similarly, $\delta_x \leftharpoondown y = \delta_{y^{-1}x}$. Hence,

$$y \twoheadrightarrow \delta_x = y \rightharpoonup \delta_x \leftharpoonup y^{-1} = \delta_{yxy^{-1}}$$

When H is finite-dim'l, the coadjoint actions make $H^{*,cop}$ into a left H-module coalgebra, and H into a right $H^{*,cop}$-module algebra. That is,

$$\Delta^{*,cop}(h \twoheadrightarrow f) = \sum (h_1 \twoheadrightarrow f_2) \otimes (h_2 \twoheadrightarrow f_1)$$

$\Delta^*(f) = \sum f_1 \otimes f_2$, $\Delta^{*,cop}(f) = \sum f_2 \otimes f_1$, $\Delta^*(f)(x \otimes y) = f(xy)$

$\Delta^{*,cop}(f)(x \otimes y) = f(yx)$

and

$$\Delta(h \twoheadleftarrow f) = \sum (h_1 \twoheadleftarrow f_2) \otimes (h_2 \twoheadleftarrow f_1).$$

$\Delta^{*,cop}(h \twoheadrightarrow f)(x \otimes y) = (h \twoheadrightarrow f)(yx)$

$= \sum \langle h_1 \rightharpoonup f \leftharpoonup \bar{S}(h_2), yx \rangle$

$= \sum f, \bar{S}(h_2) yx h_1 \rangle$

$\left(\sum (h_1 \twoheadrightarrow f_2) \otimes (h_2 \twoheadrightarrow f_1) \right)(x \otimes y)$

$= \sum \langle (h_1)_1 \rightharpoonup f_2 \leftharpoonup \bar{S}((h_1)_2), x \rangle \langle (h_2)_1 \rightharpoonup f_1 \leftharpoonup \bar{S}((h_2)_2), y \rangle$

$= \sum \langle f_2, \bar{S}(h_2) x h_1 \rangle \langle f_1, \bar{S}(h_4) y h_3 \rangle$

$= \sum \langle f, \bar{S}(h_4) y \underbrace{h_3 \bar{S}(h_2)} x h_1 \rangle = \sum \langle f, \bar{S}(h_2) yx h_1 \rangle.$

$\bar{S}(h_3) y \underbrace{\varepsilon(h_2)} x h_1$

$\bar{S}(h_2) yx h_1$