Lecture 15

Last time : $h \rightharpoondown f = \sum h_1 \rightharpoonup f \leftharpoonup \bar{S}(h_2)$, $\bar{S} = S^{-1}$ $\qquad H \curvearrowright H^*$

$f \leftharpoondown h = \sum \bar{S}(h_1) \rightharpoonup f \leftharpoonup h_2$ $\qquad H^* \curvearrowleft H$

$H^{*,cop}$ (when $\dim(H) < \infty$) has coproduct $\Delta^{*,cop}(f) = \sum f_2 \otimes f_1$

$S^{*,cop} = \bar{S}^*$

THM 3.3 (Drinfeld) Let H be a finite-dimensional Hopf algebra. There exists a unique Hopf algebra structure on the vector space

$$D(H) := H^{*,cop} \bowtie H := H^{*,cop} \otimes H$$

\bowtie

such that for all $\varphi, \psi \in H^*$ and $a, b \in H$, the following conditions are satisfied.

- The multiplication on $D(H)$ satisfies

$$(\varphi \bowtie a)\cdot(\psi \bowtie b) = \sum \varphi\,(a_1 \rightharpoondown \psi_2) \bowtie (a_2 \leftharpoondown \psi_1)\, b$$

mult in H^* $\qquad$ mult. in H.

- The unit on $D(H)$ is given by $1_{D(H)} = 1_{H^*} \bowtie 1_H = \varepsilon \bowtie 1_H$
- As a coalgebra, $D(H)$ is the tensor product of $H^{*,cop}$ and H, that is,

$$\Delta_{D(H)}(\varphi \bowtie a) = \sum (\varphi_2 \bowtie a_1) \otimes (\varphi_1 \bowtie a_2) \quad \in D(H) \otimes D(H)$$

and $\quad \varepsilon_{D(H)} = \varepsilon_{H^*} \otimes \varepsilon_H = 1_H \otimes \varepsilon$

- The antipode of $D(H)$ satisfies

$$S_{D(H)}(\varphi \bowtie a) = (\varepsilon \bowtie S(a)) \cdot (\bar{S}^*(\varphi) \bowtie 1_H)$$

$$= \sum \left(S(a_2) \rightharpoondown \bar{S}^*(\varphi_1)\right) \bowtie \left(S(a_1) \leftharpoondown \bar{S}^*(\varphi_2)\right)$$

Moreover, if $\{e_i\}$ is a basis for H, and $\{e^i\}$ is the corresponding dual basis for H^*, then $(D(H), R)$ is a quasi-triangular Hopf algebra, where

$$R = \sum_i (\varepsilon \bowtie e_i) \otimes (e^i \bowtie 1_H) \quad \in D(H) \otimes D(H)$$

whose inverse is $\bar{R} = \sum_i (\varepsilon \bowtie e_i) \otimes (S^*(e^i) \otimes 1_H) \in D(H) \otimes D(H)$.

<u>DEF</u>. Let H be a finite-dim'l Hopf algebra. The quantum double of H is the quasi-triangular Hopf algebra $D(H)$ described in THM 3.3.

<u>Note</u>. The proof of THM 3.3 involves massive calculations, which are omitted. Details can be found in Kassel's book, Majid (Physics for algebraists). Here, we illustrate a proof of $(\Delta \otimes id)(R) = R^{13} R^{23}$ as an example.

$(e^i \bowtie 1_H)(e^j \bowtie 1_H)$

By def, we need to show

$$\sum (\varepsilon \bowtie e_{i(1)}) \otimes (\varepsilon \bowtie e_{i(2)}) \otimes (e^i \bowtie 1_H) = \sum (\varepsilon \bowtie e_i) \otimes (\varepsilon \bowtie e_j) \otimes (e^i e^j \bowtie 1_H)$$

Evaluate both sides at $(a \bowtie r) \otimes (b \bowtie s) \otimes (c \bowtie t) \in (H \otimes H^*)^{\otimes 3}$.

On the one hand,

$$\sum \langle LHS, (a \bowtie r) \otimes (b \bowtie s) \otimes (c \bowtie t) \rangle$$

$$= \sum \varepsilon(a)\, \varepsilon(b)\, t(1_H)\, r(e_{i(1)})\, s(e_{i(2)})\, e^i(c)$$

$$= \sum \varepsilon(a)\, \varepsilon(b)\, t(1_H)\, \langle r \otimes s, \; e^i(c)\, e_{i(1)} \otimes e_{i(2)} \rangle$$

Since $c = \sum e^i(c)\, e_i$, so $\sum c_{(1)} \otimes c_{(2)} = \Delta(c) = \sum_{i, (e_i)} e^i(c)\, e_{i(1)} \otimes e_{i(2)}$

$$= \varepsilon(a)\, \varepsilon(b)\, t(1_H)\, rs(c)$$

On the other hand,

$$\sum_i \langle RHS, (a \bowtie r) \otimes (b \bowtie s) \otimes (c \bowtie t) \rangle$$

$$= \sum_i \varepsilon(a)\, \varepsilon(b)\, t(1_H)\, r(e_i)\, s(e_j)\, e^i e^j(c)$$

Since $\sum_i r(e_i) e^i = r$, $\sum_j s(e_j) e^j = s$

$$= \varepsilon(a)\, \varepsilon(b)\, t(1_H)\, rs(c)$$

<u>RMK</u>. By def, we have embeddings

$$\begin{tikzcd} & \cong \varepsilon \bowtie H \arrow[d] \\ H \arrow[r, hook] & D(H) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H^{*,cop} & \bowtie 1_H \arrow[d, hook] \\ H^{*,cop} \arrow[u, phantom, "\cong"] \arrow[r, hook] & D(H) \end{tikzcd}$$

Moreover, $(\varphi \bowtie a) = (\varphi \bowtie 1_H) \cdot (\varepsilon \bowtie a)$ (compare w/ $(\varepsilon \bowtie a) \cdot (\varphi \bowtie 1_H)$!)

$$= \sum_i \varphi(1_H \rightharpoonup \varepsilon) \bowtie (1_H \leftharpoonup \varepsilon)\, a$$

This implies the multiplication is completely determined by $(\varepsilon \bowtie e_i) \cdot (e^j \bowtie 1_H)$ where $\{e_i\}$, $\{e^i\}$ are dual bases as before. Indeed,

$$(e^i \bowtie e_j) \cdot (e^k \bowtie e_l) = (e^i \bowtie 1_H) \cdot \underbrace{(\varepsilon \bowtie e_j) \cdot (e^k \bowtie 1_H)}_{= \sum_i c^{jn}_{km} (e^m \bowtie 1_H) \cdot (\varepsilon \bowtie e_n)} \cdot (\varepsilon \bowtie e_l)$$

$$= \sum_i c^{jn}_{km} (e^i \bowtie 1_H)(e^m \bowtie 1_H)(\varepsilon \bowtie e_n)(\varepsilon \bowtie e_l)$$

$$= \sum_i c^{jn}_{km} (e^i e^m \bowtie e_n e_l).$$

Consequently, the universal R-matrix R defined in THM 3.3 is independent of the choice of dual basis. Indeed, R is the image of $\sum_i e_i \otimes e^i \in H \otimes H^*$ under the tensor product of the embeddings $H \hookrightarrow D(H)$, $H^* \hookrightarrow D(H)$.

Cor 3.5. Let H be a finite-dimensional Hopf algebra. Then H is a Hopf subalgebra of a QT Hopf algebra, and a Hopf algebra quotient of a CQT Hopf algebra.

PF. $H \hookrightarrow D(H)$ as Hopf subalgebra. $H^* \hookrightarrow D(H^*)$, so $(D(H^*))^* \to (H^*)^* \cong H$ is surjective. But $D(H^*)^*$ is CQT.

Example. Let $H = \mathbb{k}G$ for G a finite group w/ the canonical dual basis G for H and $\{\delta_g \mid g \in G\}$ for H^*. Note that for $H^{*,cop}$, we have $\Delta^{*,cop}(\delta_g) = \sum_{xy=g} \delta_y \otimes \delta_x$, $\varepsilon_{H^*}(\delta_g) = \delta_{g,e}$, and $\bar{S}^*(\delta_g) = \delta_{g^{-1}}$.

As a vector space, $D(H) = \mathrm{span}_{\mathbb{k}}\{\delta_g \bowtie h \mid g, h \in G\}$, and the multiplication is determined by $(\varepsilon \bowtie g) \cdot (\delta_h \bowtie e) = \delta_{ghg^{-1}} \bowtie g$

Verification is left as an exercise (what is $g \leftharpoonup \delta_x$?)

We have the following simple observation.

LEM 3.7. Suppose $\dim(H) < \infty$. Then for any $\varphi \in H^*$, $h \in H$,

$$h \rightharpoonup \varphi = \sum \langle \varphi_3 \cdot \bar{S}^*(\varphi_1), h \rangle \, \varphi_2$$

$$h \leftharpoonup \varphi = \sum \langle \varphi, \bar{S}(h_3) \cdot h_1 \rangle \, h_2$$

PF. We prove the second equality. $\forall \psi \in H^*$,

$$\langle \psi, h \leftharpoonup \varphi \rangle = \sum \langle \psi, \bar{S}^*(\varphi_1) \rightharpoonup h \leftharpoonup \varphi_2 \rangle$$

$$= \sum \langle \varphi_2, h_1 \rangle \langle \psi, h_2 \rangle \langle \bar{S}^*(\varphi_1), h_3 \rangle = \sum \langle \varphi_1, \bar{S}(h_3) \rangle \langle \varphi_2, h_1 \rangle \langle \psi, h_2 \rangle$$

$$= \sum \langle \varphi, \bar{S}(h_3) \cdot h_1 \rangle \langle \psi, h_2 \rangle = \langle \psi, \sum \langle \varphi, \bar{S}(h_3) h_1 \rangle h_2 \rangle \qquad \blacksquare$$

LEM 3.8. Suppose $\dim(H) < \infty$. Then for all $\varphi, \psi \in H^*$ and $h, k \in H$,

$$(\varphi \bowtie h)(\psi \bowtie k) = \sum \varphi(h_1 \rightharpoonup \psi \leftharpoonup \bar{S}(h_3)) \bowtie h_2 k$$

$$(\varphi \bowtie h)(\psi \bowtie k) = \sum \varphi \psi_2 \bowtie (\bar{S}^*(\psi_1) \rightharpoonup h \leftharpoonup \psi_3) k.$$

Pf. $(\varphi \bowtie h)(\psi \bowtie k) = \sum \varphi(h_1 \rightharpoonup \psi_{2}) \bowtie (h_2 \leftharpoonup \psi_{1}) k$ w.r.t. Δ^*

$$= \sum \varphi(h_1 \rightharpoonup \psi_2) \bowtie \langle \psi_1, \bar{S}((h_2)_3) \cdot (h_2)_1 \rangle (h_2)_2 k$$

$\Delta^{cop}(x) = \sum x_{(2)} \otimes x_{(1)} = \sum x_{(cop,1)} \otimes x_{(cop,2)}$

$$= \sum \varphi(h_1 \rightharpoonup \psi_2) \bowtie \langle \psi_1, \bar{S}(h_4) \cdot h_2 \rangle h_3 k$$

$$= \sum \varphi(h_1 \rightharpoonup \langle \psi_1, \bar{S}(h_4) \cdot h_2 \rangle \psi_2) \bowtie h_3 k$$

Recall $\forall f \in H^*$, $a, b \in H$, $\langle f \leftharpoonup a, b \rangle = \langle f, ab \rangle = \sum \langle f_1, a \rangle \langle f_2, b \rangle$
$= \langle \sum \langle f_1, a \rangle f_2, b \rangle$.

$$= \sum \varphi(h_1 \rightharpoonup (\psi \leftharpoonup (\bar{S}(h_4) \cdot h_2))) \bowtie h_3 k$$

$$= \sum \varphi((h_1)_1 \rightharpoonup (\psi \leftharpoonup \bar{S}(h_4) h_2) \leftharpoonup \bar{S}((h_1)_2)) \bowtie h_3 k$$

$$= \sum \varphi(h_1 \rightharpoonup (\psi \leftharpoonup \bar{S}(h_5) h_3) \leftharpoonup \bar{S}(h_2)) \bowtie h_4 k$$

$$= \sum \varphi(h_1 \rightharpoonup \psi \leftharpoonup \bar{S}(h_3)) \bowtie h_2 k. \quad \blacksquare$$

Therefore, by Rmk above, the multiplication in $D(H)$ is actually determined by

$$(\varepsilon \bowtie h)(\varphi \bowtie 1_H) = \sum (h_1 \rightharpoonup \varphi \leftharpoonup \bar{S}(h_3)) \bowtie h_2$$

THM 3.9 (Radford) Suppose $\dim(H) < \infty$. Choose $0 \neq \Lambda \in \int_H^R$ and $0 \neq \lambda \in \int_{H^*}^L$, then $\lambda \bowtie \Lambda$ is a left and right integral for $D(H)$. In particular, $D(H)$ is unimodular.

Sketch. Let $g \in H$ be the distinguished group like element. Recall by Prop 2.4.7, $\{\bar{S}(\Lambda_2), \Lambda_1\}$ and $\{S(\Lambda_1)g^{-1}, \Lambda_2\}$ are both dual bases w/ r.t. the same bilinear form, then

$$(*) \qquad \sum_i \bar{S}(\Lambda_2) \otimes \Lambda_1 = \sum_i S(\Lambda_1) g^{-1} \otimes \Lambda_2$$

Consequently, we have $\sum \Lambda_1 \otimes \Lambda_2 = \sum_i \Lambda_2 \otimes g S^2(\Lambda_1)$. and so

$\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3$ $\qquad$ $\Lambda_2 \otimes \Lambda_3 \otimes g S^2(\Lambda_1)$

$\bar{S}(\Lambda_3) g \otimes \Lambda_1 \otimes \Lambda_2 \leftarrow \Lambda_3 \otimes \Lambda_1 \otimes \Lambda_2$ $\qquad$ $g S^2(\Lambda_1) \otimes \Lambda_2 \otimes \Lambda_3 \longrightarrow (S(\Lambda_1) g^{-1}) \cdot g \otimes \Lambda_2 \otimes \Lambda_3$

$$\sum_i \bar{S}(\Lambda_3) g \Lambda_1 \otimes \Lambda_2 = \sum_i S(\Lambda_1)\Lambda_2 \otimes \Lambda_3$$
$$= 1_H \otimes \Lambda$$

By Lem 3.8,

$$(\lambda \bowtie \Lambda) \cdot (\varphi \bowtie h) = \sum_i \lambda(\Lambda_1 \rightharpoonup \varphi \leftharpoonup \bar{S}(\Lambda_3)) \bowtie \Lambda_2 h$$

$= \sum_i \lambda \langle \Lambda_1 \rightharpoonup \varphi \leftharpoonup \bar{S}(\Lambda_3), g \rangle \bowtie \Lambda_2 h$ (by def of $\int_{H^*}^L$ and g)

$= \sum_i' \lambda \langle \varphi, \bar{S}(\Lambda_3) g \Lambda_1 \rangle \bowtie \Lambda_2 h.$

$= \sum_i \lambda \langle \varphi, 1_H \rangle \bowtie \Lambda h = \varphi(1_H) \cdot \varepsilon(h) \cdot (\lambda \bowtie \Lambda) = \varepsilon_{D(H)}(\varphi \bowtie h) \cdot (\lambda \bowtie \Lambda).$

$\Rightarrow \lambda \bowtie \Lambda \in \int_{D(H)}^R$.

To show $\lambda \bowtie \Lambda \in \int_{D(H)}^L$, use $S(\int_H^L) = \int_H^R$ to derive the left integral version of (*), i.e., $\sum \Lambda_2^L \otimes \Lambda_1^L = \sum_i \Lambda_1^L \otimes S^2(\Lambda_2^L) g^{-1}$. for $0 \neq \Lambda^L \in \int_H^L$.

$\square$

Note: The g in our course is the inverse of that in Radford '94.

Cor. Suppose $\dim(H) < \infty$. TFAE.

- $D(H)$ is semisimple.
- H and H^* are s.s.
- H and H^* are co-s.s.
- $D(H)$ is co-s.s.

Moreover, if $\mathrm{char}(\Bbbk) = 0$, then TFAE

- $D(H)$ is s.s. • H is s.s. • H^* is s.s. • $S^2 = id$.
- $D(H)$ is co-s.s • H is co-s.s • H^* is co-s.s. ▨

§4. Yetter-Drinfeld modules.

Goal: describe left $D(H)$-modules in terms of H.

Given any left $D(H)$-module V w/ action $\diamondsuit$, it is automatically a left H-module

$$h \bullet v := (\varepsilon \bowtie h) \diamondsuit v$$

and a left H^*-module $\rightsquigarrow$ gives right H-comodule structure.

$$\varphi \,\square\, v := (\varphi \bowtie 1_H) \diamondsuit v.$$

They are compatible in the following sense:

$$h \bullet (\varphi \,\square\, v) = [(\varepsilon \bowtie h) \cdot (\varphi \bowtie 1_H)] \diamondsuit v$$

$$= \left(\sum (h_1 \rightharpoonup \varphi \leftharpoonup \bar{S}(h_3)) \bowtie h_2\right) \diamondsuit v = \sum (h_1 \rightharpoonup \varphi \leftharpoonup \bar{S}(h_3)) \,\square\, (h_2 \bullet v)$$

Point: " $D(H)$-mod V $\longleftrightarrow$ $V =$ H-mod $=$ H-comod + compatibility "