Lecture 16

Last time : quantum double of Hopf algebra (fin. dim'l)

$$D(H) = H^{*,\,cop} \bowtie H \qquad \text{QTHA}$$

$$(\varphi \bowtie 1_H)(\varepsilon \bowtie h) = \varphi \bowtie h \quad , \quad (\varepsilon \bowtie h)(\varphi \bowtie 1_H) = \sum (h_1 \rightharpoonup \varphi \leftharpoonup \bar{S}(h_3)) \bowtie h_2$$

§4. Yetter - Drinfeld modules.

If $V \in \mathrm{Rep}(D(H))$, then it is an H- and H^*- module. w/ compatibility

(*) $\quad h \bullet (\varphi \,\square\, v) = [(\varepsilon \bowtie h)(\varphi \bowtie 1_H)] \diamond v = \sum (h_1 \rightharpoonup \varphi \leftharpoonup \bar{S}(h_3)) \,\square\, (h_2 \bullet v)$

<u>DEF</u>. Let H be a finite-dim'l Hopf algebra. A (left-right-) Yetter - Drinfeld module of H is a triple (V, γ, ρ) where V is a finite-dim'l v.sp., $\gamma : H \otimes V \to V$, $\rho : V \to V \otimes H$ are linear maps such that

- (V, γ) is a left H-module
- (V, ρ) is a right H-comodule.
- this diagram commutes.

$$\begin{tikzcd}
H \otimes V \arrow[r, "\Delta \otimes id"] \arrow[d, "\Delta \otimes \rho"'] & H \otimes H \otimes V \arrow[d, "id \otimes \gamma"] \\
H \otimes H \otimes V \otimes H \arrow[d, "id \otimes \tau \otimes id"'] \arrow[r, phantom, "\circlearrowright"] & H \otimes V \arrow[d, "\tau"] \\
H \otimes V \otimes H \otimes H \arrow[d, "\gamma \otimes \mu"'] & V \otimes H \arrow[d, "\rho \otimes id"] \\
V \otimes H & V \otimes H \otimes H \arrow[l, "id \otimes \mu"]
\end{tikzcd}$$

The category of Yetter - Drinfeld modules of H is denoted by ${}_H\mathcal{YD}^H$.

In sigma notation, we have

$$(**)\quad \sum (h_1 \bullet v_0) \otimes h_2 v_1 = \sum (h_2 \bullet v)_0 \otimes (h_2 \bullet v)_1 h_1$$

for all $h \in H$, $v \in V$.

THM. Suppose $\dim(H) < \infty$. Any left $D(H)$-module has a natural structure of a YD-module, and vice versa.

Pf. Assume $V \in \mathrm{Rep}(D(H))$. By previous discussions, $H \overset{\bullet}{\curvearrowright} V$, $H^* \overset{\square}{\curvearrowright} V$.

By LEM 1.5.4, V has a natural right H-comodule structure: if $\{e_i, e^i\}$ is a dual basis for H and H^*, then the coaction is defined by

$$\rho : V \to V \otimes H, \quad \rho(v) := \sum_i (e^i \square v) \otimes e_i = \sum_{(v)} v_{(0)} \otimes v_{(1)}$$

We will show $(V, \bullet, \rho) \in {}_H YD^H$ by verifying $(**)$. $\forall\, h \in H, v \in V, \varphi \in H^*$,

$$(id \otimes \varphi)\Big(\sum (h_{(1)} \bullet v_{(0)}) \otimes h_{(2)} v_{(1)}\Big) = (id \otimes \varphi)\Big(\sum (h_{(1)} \bullet (e^i \square v)) \otimes h_{(2)} e_i\Big)$$

$$= \sum \varphi_{(1)}(e_i)\, \varphi_{(2)}(h_{(2)}) \big(h_{(1)} \bullet (e^i \square v)\big)$$

$$= \sum \varphi_{(2)}(h_{(2)}) \big(h_{(1)} \bullet (\varphi_{(1)} \square v)\big)$$

$$= \sum \varphi_{(2)}(h_{(4)}) \big(h_{(1)} \rightharpoonup \varphi_{(1)} \leftharpoonup \bar{S}(h_{(3)})\big) \square (h_{(2)} \bullet v)$$

Recall $\sum \psi_{(2)}(a)\psi_{(1)} = \psi \leftharpoonup a$ and $\sum \psi_{(1)}(a)\psi_{(2)} = a \rightharpoonup \psi$ $\forall\, \psi \in H^*, a \in H$

so $$LHS = \sum \big(h_{(1)} \rightharpoonup \varphi \leftharpoonup h_{(4)} \bar{S}(h_{(3)})\big) \square (h_{(2)} \bullet v)$$

$$= \sum (h_{(1)} \rightharpoonup \varphi) \square (h_{(2)} \bullet v)$$

$$= \sum \varphi_{(1)}(h_{(1)})\, \varphi_{(2)} \square (h_{(2)} \bullet v) = \sum \boxed{\varphi_{(1)}(h_{(1)})\, \varphi_{(2)}(e_i)}\, e^i \square (h_{(2)} \bullet v)$$

$$= \sum \left(e^i \square (h_{(2)} \bullet v)\right) \varphi(e_i h_{(1)})$$

$$= (id \otimes \varphi) \sum (e^i \square (h_{(2)} \bullet v)) \otimes e_i h_{(1)}$$

Since φ is arbitrary, $(**)$ holds, i.e., $(V, \bullet, \rho) \in {}_H\mathcal{YD}^H$.

Conversely, suppose $(V, \bullet, \rho) \in {}_H\mathcal{YD}^H$, we have $H^* \curvearrowright V$ by

$$\varphi \square v := \sum v_{(0)} \cdot \varphi(v_{(1)}), \quad \forall \varphi \in H^*, v \in V.$$

Define $(\varphi \bowtie h) \diamond v := \varphi \square (h \bullet v)$. To show this is an action, it suffices to verify $(*)$. $\forall \varphi \in H^*, v \in V$,

$$\sum \left(h_{(1)} \rightharpoonup \varphi \leftharpoonup \bar{S}(h_{(3)})\right) \square (h_{(2)} \bullet v)$$

$$= \sum (h_{(2)} \bullet v)_{(0)} \langle h_{(1)} \rightharpoonup \varphi \leftharpoonup \bar{S}(h_{(3)}), (h_{(2)} \bullet v)_{(1)} \rangle$$

$$= \sum (h_{(2)} \bullet v)_{(0)} \langle \varphi \leftharpoonup \bar{S}(h_{(3)}), (h_{(2)} \bullet v)_{(1)} h_{(1)} \rangle$$

by $(**)$
$$= \sum (h_{(1)} \bullet v_{(0)}) \langle \varphi \leftharpoonup \bar{S}(h_{(3)}), h_{(2)} v_{(1)} \rangle$$

$$= \sum \left(h \bullet v_{(0)}\right) \langle \varphi, v_{(1)} \rangle = h \bullet (\varphi \square v).$$

i.e., $(V, \diamond) \in \mathrm{Rep}(D(H))$. ∎

It is easy to derive that $\mathrm{Rep}(D(H)) \cong {}_H\mathcal{YD}^H$ as categories.

§5. Categorical perspective of the Drinfeld double.

Def. A monoidal category is a 6-tuple $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ such that

- $\mathcal{C}$ is a category, $\mathbb{1} \in Ob(\mathcal{C})$, and $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor.
  ↳ abstract "binary operation" on category $\mathcal{C}$.
- $l : \mathbb{1} \otimes - \to id_{\mathcal{C}}$, $r : - \otimes \mathbb{1} \to id_{\mathcal{C}}$, $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$

  are natural isomorphisms.

· $\forall\, a, b, c, d \in Ob(\mathcal{C})$, the following diagrams are commutative:

$$\begin{tikzcd}
 & ((a\otimes b)\otimes c)\otimes d \arrow[dl, "\alpha_{a\otimes b, c, d}"'] \arrow[dr, "\alpha_{a,b,c}\otimes d\ (\nearrow id_d)"] & \\
(a\otimes b)\otimes(c\otimes d) \arrow[d, "\alpha_{a,b,c\otimes d}"'] & & (a\otimes(b\otimes c))\otimes d \arrow[d, "\alpha_{a,b\otimes c,d}"] \\
a\otimes(b\otimes(c\otimes d)) & & a\otimes((b\otimes c)\otimes d) \arrow[ll, "a\otimes\alpha_{b,c,d}"']
\end{tikzcd}$$

"pentagon axiom"

$$\begin{tikzcd}
(a\otimes \mathbb{1})\otimes b \arrow[rr, "\alpha_{a,\mathbb{1},b}"] \arrow[dr, "r_a\otimes b"'] & & a\otimes(\mathbb{1}\otimes b) \arrow[dl, "a\otimes \ell_b"] \\
 & a\otimes b &
\end{tikzcd}$$

"triangle axiom"

[MacLane, GTM 5].

A monoidal category $\mathcal{C}$ is strict if α, ℓ, r are all identities.

Rmk. MacLane's coherence theorem implies that any monoidal category is equivalent to a strict one.

Example. For any bialgebra B (over $\Bbbk$), $Rep(B)$ is a monoidal category in the following way:

$\otimes = \otimes_{\Bbbk}$, $\mathbb{1} = (\Bbbk, \varepsilon)$, $\forall (V,\rho) \in Rep(B)$, $\ell_{(V,\rho)} : (\Bbbk,\varepsilon)\otimes_{\Bbbk}(V,\rho) \to (V,\rho)$, $\alpha\otimes_{\Bbbk} v \mapsto \alpha v$, need to show: $\ell_{(V,\rho)}$ is a B-module map.

$$\ell_{(V,\rho)}(b\cdot(\alpha\otimes v)) = \ell_{(V,\rho)}\left(\sum_{(1)} \varepsilon(b_{(1)})\alpha \otimes_{\Bbbk} b_{(2)}\cdot v\right)$$
$$= \sum_{(1)} \alpha\, \varepsilon(b_{(1)})(b_{(2)}\cdot v) = \alpha(b\cdot v) = b\cdot(\alpha v).$$

counit condition

$\alpha_{U,V,W} : (U \otimes_{\Bbbk} V) \otimes_{\Bbbk} W \to U \otimes_{\Bbbk} (V \otimes_{\Bbbk} W)$ is defined to be the canonical $\Bbbk$-linear map. Need (1) show it is a B-module map

(2) it satisfies the pentagon axiom. (Use coassociativity).

Example. $\mathrm{Vec}_{\Bbbk}$: the category of fin dim'l v. sp. $/\Bbbk$.

$\otimes = \otimes_{\Bbbk}$, $\mathbb{1} = \Bbbk$, α, ℓ, r as above. ($\triangle$ and $\pentagon$ are obviously true).

"tensor category" sometimes implies certain linearity is imposed.

"commutativity of monoidal category" can be measured by braidings.

Let $\bar{\otimes}$ be the reversed tensor product on $\mathscr{C}$, i.e., $a \,\bar{\otimes}\, b := b \otimes a$.

DEF. A **braiding** on a monoidal category $\mathscr{C}$ is a natural transformation $\beta : - \otimes - \to - \,\bar{\otimes}\, -$ satisfying the following hexagon axioms:

$$\begin{tikzcd}
(a \otimes b) \otimes c \arrow[r, "\beta_{a\otimes b, c}"] \arrow[d, "\alpha_{a,b,c}"'] & c \otimes (a \otimes b) \\
a \otimes (b \otimes c) \arrow[d, "a \otimes \beta_{b,c}"'] & (c \otimes a) \otimes b \arrow[u, "\alpha_{c,a,b}"'] \\
a \otimes (c \otimes b) \arrow[r, "\alpha^{-1}_{a,c,b}"] & (a \otimes c) \otimes b \arrow[u, "\beta_{a,c} \otimes b"']
\end{tikzcd}
\qquad
\begin{tikzcd}
a \otimes (b \otimes c) \arrow[r] \arrow[d] & (b \otimes c) \otimes a \\
(a \otimes b) \otimes c \arrow[d] & b \otimes (c \otimes a) \arrow[u] \\
(b \otimes a) \otimes c \arrow[r] & b \otimes (a \otimes c) \arrow[u]
\end{tikzcd}$$

for all $a, b, c \in \mathrm{Ob}(\mathscr{C})$. We call $(\mathscr{C}, \beta)$ a **braided monoidal category**.

Prop. Let B be a bialgebra. There exists a braiding on the monoidal category $\mathrm{Rep}(B)$ if and only if there exists $R \in B \otimes B$ s.t. (B, R) is QT.

Sketch. Suppose (B, R) is QT. For any $V, W \in \mathrm{Rep}(B)$, define

$\beta_{V,W} : V \otimes W \to W \otimes V$ by $\beta_{V,W}(v \otimes w) := \tau(R \cdot (v \otimes w))$.

By Lem 1.2, $\beta_{V,W}$ is an isom of B-modules. Then it is easy to check QT condition $((\Delta \otimes \mathrm{id})R = R^{13}R^{23}, \ \cdots)$

Conversely, suppose $\mathrm{Rep}(B)$ has a braiding β. Define

$$R := \tau(\beta_{B,B}(1_B \otimes 1_B)) \in B \otimes B.$$

- $R \in (B \otimes B)^{\times}$ ✓
- Almost cocomm?

  $\forall V \in \mathrm{Rep}(B)$, $v \in V$, define $\varphi_v : B \to V$ by $\varphi_v(h) := h \cdot v$.

  (this is B-linear : $h \cdot \varphi_v(1_B) = h \cdot v = \varphi_v(h)$)

  The naturality of β implies $\forall V, W \in \mathrm{Rep}(B)$, $\forall v \in V$, $w \in W$,

$$\begin{tikzcd}
1_B \otimes 1_B \arrow[d, mapsto] & B \otimes B \arrow[r, "\beta_{B,B}"] \arrow[d, "\varphi_v \otimes \varphi_w"'] \arrow[dr, phantom, "\circlearrowright"] & B \otimes B \arrow[d, "\varphi_w \otimes \varphi_v"] \\
{} & V \otimes W \arrow[r, "\beta_{V,W}"] & W \otimes V
\end{tikzcd}$$

$$\begin{aligned}
\beta_{V,W}(v \otimes w) &= (\varphi_w \otimes \varphi_v)(\beta_{B,B}(1_B \otimes 1_B)) \\
&= \tau(\varphi_V \otimes \varphi_W)\big(\underbrace{\tau(\beta_{B,B}(1_B \otimes 1_B))}\big) \\
&= \tau((\varphi_V \otimes \varphi_W)R) \underset{\substack{\uparrow \\ \varphi_\cdot \text{ is } B\text{-linear}}}{=} \tau(R(v \otimes w))
\end{aligned}$$

$$\Rightarrow \quad \underset{\substack{\| \ \ \beta_{B,B} \text{ is } B\text{-linear} \\ \Delta(h) \cdot \beta_{B,B}(1_B \otimes 1_B)}}{\beta_{B,B}(\Delta(h))} = \tau(R(\Delta(h))). \quad \Rightarrow \tau\Delta(h) \cdot R = R \cdot \Delta(h) \quad ✓$$

- QT condition is implied by ⬡. ☒

In particular, for any finite-dim'l Hopf algebra H, $\mathrm{Rep}(D(H))$ is a braided monoidal category.

Similar to the quantum double construction for Hopf algebras, one can contruct a braid monoidal category from a monoidal cat.
"Drinfeld center construction". Joyal - Street, Majid, (Drinfeld, unpublished)

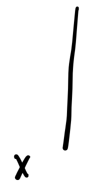
<u>DEF</u>. Let $\mathcal{C}$ be a strict monoidal category. The Drinfeld center of $\mathcal{C}$, denoted by $\mathcal{Z}(\mathcal{C})$, is a category whose objects are pairs $(V, c_{-,V})$ where $V \in Ob(\mathcal{C})$, $c_{-,V}$ is a natural isom. $c_{X,V} : X \otimes V \xrightarrow{\cong} V \otimes X$ (natural in X), called a half-braiding on V, s.t. for all $X, Y \in Ob(\mathcal{C})$,

$$c_{X\otimes Y, V} = (c_{X,V} \otimes id_Y)(id_X \otimes c_{Y,V})$$

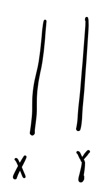
A morphism $(V, c_{-,V})$ to $(W, c_{-,W})$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f : V \to W$ in $\mathcal{C}$ s.t. $\forall X \in Ob(\mathcal{C})$,

$$(f \otimes id_X)\, c_{X,V} = c_{X,W}\, (id_X \otimes f).$$

A graphical calculus on monoidal categories:



$\rightsquigarrow$ $X \in Ob(\mathcal{C})$ or $id_X : X \to X$,

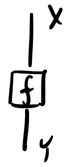


$\rightsquigarrow X \otimes Y$.

$\rightsquigarrow (X \otimes Y) \otimes Z$.

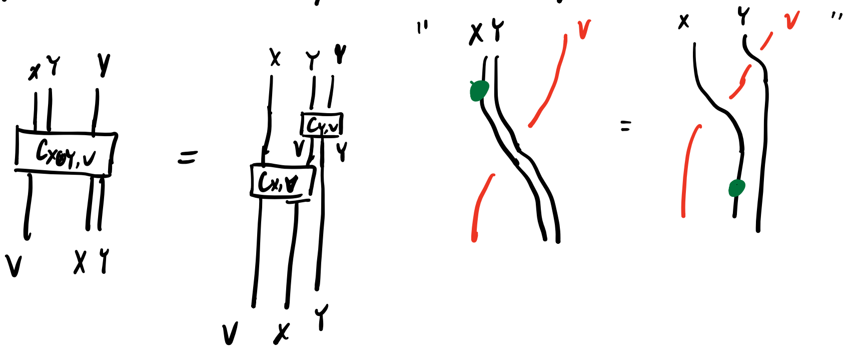
if strict:

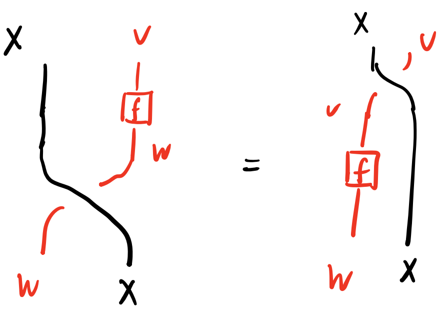
$\rightsquigarrow X \otimes Y \otimes Z$



$\rightsquigarrow f : X \to Y.$

$$c_{X\otimes Y, V} = (c_{X,V} \otimes id_Y)(id_X \otimes c_{Y,V})$$

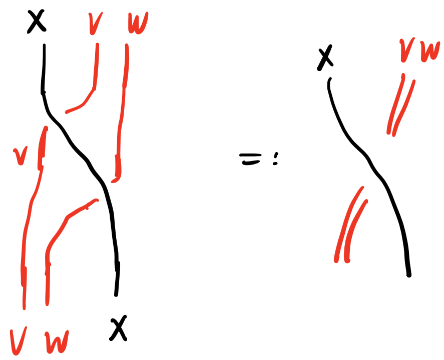




$Z(\mathcal{C})$: "central object + how they commute w/ others"

<u>Fact</u>: • $Z(\mathcal{C})$ is a braided monoidal category w/

$$(V, c_{-,V}) \otimes_{Z(\mathcal{C})} (W, c_{-,W}) = (V \otimes_{\mathcal{C}} W, \; c_{-,V\otimes W})$$



and braiding : $\beta_{(V,c_{-,V}),(W,c_{-,W})} := c_{V,W}$

- $Z(\mathrm{Rep}(H))$ for $\dim(H) < \infty$.

  A half-braiding $c_{-,V}$ on $V \in \mathrm{Rep}(H)$ endow V w/ a coaction $\rho : V \xrightarrow{1_H \otimes -} H \otimes V \xrightarrow{c_{H,V}} V \otimes H$,

  $\rho(v) := c_{H,V}(1_H \otimes v)$

  Then $(V, \cdot, \rho)$ can be shown to be a YD-module

__THM__. Let H be a finite-dim'l Hopf algebra over a field $\mathbb{k}$. Then

$$Z(\mathrm{Rep}(H)) \cong \mathrm{Rep}(D(H)) \cong {}_H\mathcal{YD}^H$$ as braided monoidal categories.

For a detailed proof, see Kassel's book.