Lecture 2.

Notes : BIMSA course webpage. (After each lecture).

Last time : algebra (A, μ, η)

coalgebra (C, Δ, ε)

$\Bbbk G$, $U(\mathfrak{g})$

2. The sigma notation

Let C be a coalgebra. $\forall c \in C$, we write

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

symbolic, not indicate particular elements of C

"place holder"

<u>Example</u>. Recall $\tau : C \otimes C \to C \otimes C$ swap tensor components.

$\tau(\Delta(c)) = \sum_{(c)} c_{(2)} \otimes c_{(1)}$ in sigma notation.

C is cocommutative $\iff$ $\sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)}$, $\forall c \in C$.

sometimes omit (c),

omit "( )" : $\sum c_1 \otimes c_2$

omit "$\sum$" : $c_1 \otimes c_2$

Powerful when performing multiple Δ.

$$(id \otimes \Delta)\,\Delta(c) = (id \otimes \Delta)\Big(\sum_{(c)} c_{(1)} \otimes c_{(2)}\Big)$$

$$= \sum_{(c)} c_{(1)} \otimes \sum_{(c_{(2)})} (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}$$

$$= \sum_{(c)} c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}$$

lexical order : $(1) \leftarrow (2)(1) \leftarrow (2)(2)$

coassociative.

$$= (\Delta \otimes id)\Delta(c) = (\Delta \otimes id)\left(\sum_{(c)} c_{(1)} \otimes c_{(2)}\right)$$

$$= \sum_{(c)} (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)}$$

lexical order : $(1)(1) \leftarrow (1)(2) \leftarrow (2)$

coassociativity $\Rightarrow$ in sigma notation, only lexical order matters.

$\Rightarrow$ when writing multiple Δ in sigma notation, reorder in lexical order.

↳ Sweedler notation

$$(id \otimes \Delta)\Delta(c) = (\Delta \otimes id)\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$$

Exercises in Sweedler's book.

$$\Delta^{[2]}(c) := \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$$

Inductively define $\Delta^{[n-1]}(c) := (\Delta \otimes id_{C^{\otimes(n-2)}})\, \Delta^{[n-2]}(c)$

$$= \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes \cdots \otimes c_{(n)}$$

Recall

counit axiom : $(\varepsilon \otimes id)\Delta(c) = c$ (identify $1 \otimes c$ with c)

$\Downarrow$ sigma notation

$$(\varepsilon \otimes id)\left(\sum_{(c)} c_{(1)} \otimes c_{(2)}\right) = \boxed{\sum_{(c)} \varepsilon(c_{(1)})\, c_{(2)} = c}$$

The other counit axiom : $\sum c_{(1)}\, \varepsilon(c_{(2)}) = c . \quad \forall c \in C$

(c)

Example. $\displaystyle\sum_{(c)} c_{(1)} \otimes \underbrace{\varepsilon(c_{(3)})}_{\in \Bbbk} \otimes c_{(2)}$

$$= \sum_{(c)} c_{(1)} \otimes \varepsilon(c_{(3)})\, c_{(2)} \quad \left(= (\mathrm{id} \otimes (\mathrm{id} \otimes \varepsilon)\Delta) \sum_{(c)} c_{(1)} \otimes c_{(2)} \right)$$

$$= \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

$$= \Delta(c)$$

3. Duals (I)

For vector space V, let $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$. Evaluation gives rise to a bilinear form

$$\langle \cdot , \cdot \rangle : V^* \otimes V \longrightarrow \Bbbk$$

$$\langle f, v \rangle := f(v)$$ use them interchangeably.

If $\varphi : V \to W$ is $\Bbbk$-linear, its transpose is defined to be

$$\varphi^* : W^* \longrightarrow V^*$$

$$\langle \underbrace{\varphi^*(f)}_{\in V^*}, \underbrace{v}_{\in V} \rangle := \langle \underbrace{f}_{\in W^*}, \underbrace{\varphi(v)}_{\in W} \rangle, \quad \forall\, v \in V,\ f \in W^*.$$

Note that for any vector space V, $V^* \otimes V^* \subseteq (V \otimes V)^*$.

LEM 3.1 If (C, Δ, ε) is a coalgebra, then

$(C^*, \mu := \Delta^*|_{C^* \otimes C^*}, \eta := \varepsilon^*)$ is an algebra.

$\Delta : C \to C \otimes C, \quad \Delta^* : (C \otimes C)^* \to C^*$

Pf. $\forall c \in C$, $f, g, h \in C^*$,

$\langle \mu(\mu \otimes id)(f \otimes g \otimes h), c \rangle$ $\qquad (\mu = \Delta^*)$

$= \langle (\mu \otimes id)(f \otimes g \otimes h), \Delta(c) \rangle$

$= \sum_{(c)} \langle \mu(f \otimes g), c_{(1)} \rangle \cdot \langle h, c_{(2)} \rangle$

(alternatively, $= \langle f \otimes g \otimes h, (\Delta \otimes id)\Delta(c) \rangle$ use coassociativity and done)

$= \sum_{(c)} \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle \langle h, c_{(3)} \rangle$

$= \sum_{(c)} \langle f, c_{(1)} \rangle \langle \mu(g \otimes h), c_{(2)} \rangle = \sum_{(c)} \langle (id \otimes \mu)(f \otimes g \otimes h), c_{(1)} \otimes c_{(2)} \rangle$

$= \sum_{(c)} \langle \mu(id \otimes \mu)(f \otimes g \otimes h), c \rangle$

$\langle \mu(\eta \otimes id)(f), c \rangle = \langle (\eta \otimes id)(f), \Delta(c) \rangle = \langle f, (\varepsilon \otimes id)\Delta(c) \rangle$

$= \langle f, c \rangle = \cdots = \langle \mu(id \otimes \eta)(f), c \rangle$. $\blacksquare$

Same notation as above,

Rmk. If C is cocomm, then C^* is comm.

$\langle \mu(f \otimes g), c \rangle = \sum_{(c)} \langle f, c_{(1)} \rangle \langle g, c_{(2)} \rangle = \sum_{(c)} \langle f, c_{(2)} \rangle \langle g, c_{(1)} \rangle$

$= \langle \mu(g \otimes f), c \rangle$.

If we start w/ an algebra, when $\underset{\parallel}{A}$ is not finite-dimensional, (A, μ, η)

then $\mu^* : A^* \to (A \otimes A)^*$ may not lie in $A^* \otimes A^*$.

A proper notion of finiteness is needed.

Def. If A is an algebra, then the finite dual of A is defined to be

$$A^\circ := \{ f \in A^* \mid f(I) = 0 \text{ for some ideal } I \subseteq A \text{ of finite codim} \}.$$

(two-sided)

$\Updownarrow$

$\dim(A/I) < \infty$

Prop 3.3 If (A, μ, η) is an algebra, then $\mu^*(A^\circ) \subseteq A^\circ \otimes A^\circ$, and (A°, μ^*, η^*) is a coalgebra.

Notation: • $\forall a \in A$, $f \in A^*$, define $a \rightharpoonup f \in A^*$ and $f \leftharpoonup a \in A^*$ by

$$\langle a \rightharpoonup f, b \rangle := \langle f, ba \rangle$$
$$\langle f \leftharpoonup a, b \rangle := \langle f, ab \rangle$$

• For any vector spaces $W \subseteq V$, define $W^\perp := \{ f \in V^* \mid \langle f, W \rangle = 0 \}$.
$\forall X \subseteq V^*$, define $X^\perp := \{ v \in V \mid \langle X, v \rangle = 0 \}$
By def, $(W^\perp)^\perp = W$, and $W^\perp \cong (V/W)^*$.
The codimension of $W \subseteq V$ is $\dim(V/W)$.

(The following are equivalent)

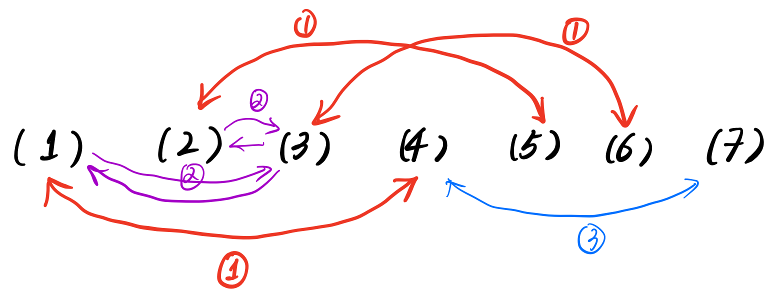
Lem 3.4 Let (A, μ, η) be an algebra. TFAE.

(1) f vanishes on a right ideal of A of finite codimension.
(2) f vanishes on a left ideal of A of finite codim.
(3) f vanishes on an ideal of A of finite codim, i.e., $f \in A^\circ$.
(two-sided)

(4) $\dim(A \rightharpoonup f) < \infty$ (5) $\dim(f \leftharpoonup A) < \infty$ (6) $\dim(A \rightharpoonup f \leftharpoonup A) < \infty$

(7) $\mu^*(f) \in A^* \otimes A^*$.

<u>Plan</u>.



<u>Sketch</u>.

(1) $\Rightarrow$ (4). Suppose $f(R) = 0$ for some right ideal $R \subseteq A$ w/ $\dim(A/R) < \infty$.

Then $\langle a \rightharpoonup f, R \rangle = \langle f, Ra \rangle = \langle f, R \rangle = 0$ $\forall a \in A$.

$\Rightarrow$ $A \rightharpoonup f \subseteq R^{\perp} \cong (A/R)^*$ finite dim'l.

(4) $\Rightarrow$ (1) Suppose $\dim(A \rightharpoonup f) < \infty$, then $R := (A \rightharpoonup f)^{\perp}$ is a right ideal of A. (To show $\forall r \in R$, $\forall a, x \in A$, $\Rightarrow rx \in R$, we need to show $\langle a \rightharpoonup f, rx \rangle = 0$. But this is clear: $\langle a \rightharpoonup f, rx \rangle = \langle xa \rightharpoonup f, r \rangle = 0$). Moreover, $\langle f, R \rangle = \langle 1_A \rightharpoonup f, R \rangle = 0$. Finally, $(A \rightharpoonup f)^* \cong (R^{\perp})^* \cong A/R$ is finite dim'l.

Similarly, (2) $\Leftrightarrow$ (5), (3) $\Leftrightarrow$ (6).

Clearly, (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Suppose $f(L) = 0$ for some left ideal $L \subseteq A$ of fin. codim. Then A/L is a fin. dimensional A-module, i.e., $\exists$ algebra homomorphism $\varphi: A \longrightarrow \mathrm{End}_{\mathbb{k}}(A/L)$. Let $I = \ker(\varphi)$, then I is a two-sided ideal of finite codim. Moreover, $I \subseteq L$, so $f(I) = 0$. Similarly, (1) $\Rightarrow$ (3). ((1) to (6) are all equiv.)

(4) $\Rightarrow$ (7). Assume $n := \dim(A \rightharpoonup f) < \infty$. Choose a basis

$\{g_1, \cdots, g_n\}$ for $A \rightharpoonup f$. Then $\forall a \in A$, $a \rightharpoonup f = \sum_{j=1}^{n} h_j(a) g_j$

for some $h_1, \cdots, h_n \in A^*$. Hence, for any $a, b \in A$,

$$\langle \mu^*(f), b \otimes a \rangle = \langle f, ba \rangle = \langle a \rightharpoonup f, b \rangle$$

$$= \sum_{j=1}^{n} \langle h_j, a \rangle \langle g_j, b \rangle = \langle \sum_{j=1}^{n} g_j \otimes h_j, b \otimes a \rangle$$

So $\mu^*(f) = \sum_{j=1}^{n} g_j \otimes h_j \in A^* \otimes A^*$.

(7) $\Rightarrow$ (4). If $\mu^*(f) = \sum_{j=1}^{n} g_j \otimes h_j \in A^* \otimes A^*$ for some $g_j, h_j \in A^*$,

then the computation above implies $A \rightharpoonup f \subseteq \operatorname{span}_{\mathbb{k}} \{g_1, \cdots, g_n\}$. $\blacksquare$

<u>Sketch Pf of Prop 3.3</u>. Let $f \in A^\circ$. By Lem 3.4, we can choose a basis $\{g_1, \cdots, g_n\}$ for $A \rightharpoonup f$. Since $g_j \in A \rightharpoonup f$ for each $1 \le j \le n$, so $A \rightharpoonup g_j \subseteq A \rightharpoonup (A \rightharpoonup f) \subseteq A \rightharpoonup f$ fin. dim'l.
$\Rightarrow$ $g_j \in A^\circ$ by Lem 3.3.

Since $\{g_1, \cdots, g_n\}$ is linearly independent, $\exists a_1, \cdots, a_n \in A$ s.t. $g_i(a_j) = \delta_{i,j}$. Then $\exists h_1, \cdots, h_n \in A^*$ s.t. $\forall b \in A$, $b \rightharpoonup f = \sum_{i=1}^{n} h_i(b) \cdot g_i$
To show $h_j \in A^\circ$, $\forall 1 \le j \le n$, note that

$$\langle f \leftharpoonup a_j, b \rangle = \langle f, a_j b \rangle = \langle \mu^*(f), a_j \otimes b \rangle$$

$$= \langle \sum_{i=1}^{n} g_i \otimes h_i, a_j \otimes b \rangle = \sum_{i=1}^{n} \underbrace{\langle g_i, a_j \rangle}_{\delta_{i,j}} \langle h_i, b \rangle = \langle h_j, b \rangle$$

$\Rightarrow h_j = f \leftharpoondown a_j \in f \leftharpoondown A \Rightarrow h_j \leftharpoondown A \subseteq f \leftharpoondown A$ fin. dim'l.

$\Rightarrow$ By LEM 3.3, $h_j \in A^\circ$.

$$\Rightarrow \mu^*(f) = \sum_{j=1}^{n} g_j \otimes h_j \in A^\circ \otimes A^\circ .$$

The rest of the proof is routine. ("dualize product and unit diagrams").

$\blacksquare$