Lecture 3

Note available at yilongwang11.github.io/teaching/22Hopf/22Hopf.html

No class on Oct 4. Make up at end of semester.

Last time: sigma notation Δ

dual A = algebra $\Rightarrow$ A° = coalgebra

§4. Bialgebra

For two algebras (A_j, μ_j, η_j), $j=1,2$, there is a natural algebra structure on $A_1 \otimes A_2$:

$$\mu: (A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \xrightarrow{id_{A_1} \otimes \tau \otimes id_{A_2}} A_1 \otimes A_1 \otimes A_2 \otimes A_2 \xrightarrow{\mu_1 \otimes \mu_2} A_1 \otimes A_2$$

and $\eta := \eta_1 \otimes \eta_2$. i.e., $(a_1 \otimes a_2)\cdot(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$ $\forall a_j, b_j \in A_j$, $j=1,2$. $1_{A_1 \otimes A_2} = 1_{A_1} \otimes 1_{A_2}$

Similarly, the tensor product of two coalgebra is also a coalgebra.

- In this course, an algebra homomorphism (alg. map) preserves the unit.

DEF. A bialgebra (over $\mathbb{k}$) is a 5-tuple $(B, \mu, \eta, \Delta, \varepsilon)$, where (B, μ, η) is an algebra, (B, Δ, ε) is a coalgebra. s.t. either of the following equivalent conditions holds:

- Δ, ε are algebra homomorphisms
- μ, η are coalgebra homomorphisms.

$\Delta: B \to \underbrace{B \otimes B}$ alg structure on tensor
$\varepsilon: B \to \mathbb{k}$

(The equivalence is left as exercise)

A subspace I of a bialgebra B is a (two-side) bi-ideal if it is both an ideal and a coideal. A $\mathbb{k}$-linear map between two bialgebras is a bialgebra homomorphism if it is both an algebra map and a coalgebra map.

Example. For any group G, $\Bbbk G$, w/ algebra / coalgebra structures defined before, is a bialgebra : $\forall g, h \in G$,

$$\Delta(gh) = gh \otimes gh = (g \otimes g) \cdot (h \otimes h) = \Delta(g) \cdot \Delta(h)$$

$$\varepsilon(gh) = 1 = \varepsilon(g)\,\varepsilon(h).$$

fundamental example of cocommutative bialgebras.

Example. For any Lie algebra $\mathfrak{g}$, $\mathcal{U}(\mathfrak{g})$ is a bialgebra w/ algebra / coalg structures above.

Prop 4.4. If $(B, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, then $(B^\circ, \Delta^*, \varepsilon^*, \mu^*, \eta^*)$ is also a bialgebra.

Sketch. By Lem 3.1, $(B^*, \Delta^*, \varepsilon^*)$ is an algebra, and by Prop 3.3, (B°, μ^*, η^*) is a coalgebra.

$B^* \otimes B^* \hookleftarrow B^\circ \otimes B^\circ$

$\downarrow$

First show : B° is a subalgebra of B^*. $(\Delta^* : (B \otimes B)^* \to B^*)$

By def, $\varepsilon^*(1) = \varepsilon$. Since $\mathrm{Ker}(\varepsilon)$ has codim 1, so $\varepsilon \in B^\circ$.

$(\varepsilon : B \to \Bbbk)$

Now, for any $f, g \in B^\circ$, we have $B \rightharpoonup f$ and $B \rightharpoonup g$ are finite-dim'l by Lem 3.4. For any $a, b \in B$, since B is bialg, we have

$$\langle a \rightharpoonup (fg), b \rangle = \langle fg, ba \rangle = \langle f \otimes g, \Delta(ba) \rangle$$

$\hookrightarrow \sum (ba)_{(1)} \otimes (ba)_{(2)}$

$= \sum b_{(1)} a_{(1)} \otimes b_{(2)} a_{(2)}$

$$= \sum \langle f, b_{(1)} a_{(1)} \rangle \langle g, b_{(2)} a_{(2)} \rangle$$

$$= \sum \langle a_{(1)} \rightharpoonup f, b_{(1)} \rangle \langle a_{(2)} \rightharpoonup g, b_{(2)} \rangle$$

$$= \sum \langle (a_{(1)} \rightharpoonup f) \otimes (a_{(2)} \rightharpoonup g), \underbrace{b_{(1)} \otimes b_{(2)}}_{\Delta(b)} \rangle$$

$$= \sum \langle (a_{(1)} \rightharpoonup f)(a_{(2)} \rightharpoonup g), b \rangle$$

$$\Rightarrow a \rightharpoonup (fg) = \sum (a_{(1)} \rightharpoonup f)(a_{(2)} \rightharpoonup g) \in \mathrm{span}_{\Bbbk}(B \rightharpoonup f)(B \rightharpoonup g)$$

$$\Rightarrow \dim(B \rightharpoonup (fg)) < \infty \Rightarrow fg = \Delta^*(f \otimes g) \in B^*$$

$\Rightarrow \Delta^*(B^\circ \otimes B^\circ) \subseteq B^\circ$, $\varepsilon^*(\Bbbk) \subseteq B^\circ \Rightarrow (B^\circ, \Delta^*, \varepsilon^*)$ is a subalgebra of $(B^*, \Delta^*, \varepsilon^*)$.

Next show : μ^*, η^* are algebra maps. $\forall f, g \in B^\circ$, $a, b \in B$.

- $\langle \mu^*(f) \cdot \mu^*(g) , a \otimes b \rangle \overset{?}{=} \langle \mu^*(fg) , a \otimes b \rangle$

(left as exercise)

$$\langle fg, \mu(a \otimes b) \rangle = \langle fg, ab \rangle$$

$$\sum \langle f_{(1)} g_{(1)} \otimes f_{(2)} g_{(2)} , a \otimes b \rangle$$

recall ... $\sum \langle f_{(1)}, x \rangle \langle f_{(2)}, y \rangle = f(xy)$, $\forall x, y \in B$.

$\blacksquare$

DEF. Let C be a coalgebra.

- An element $x \in C$ is **group-like** if $\Delta(x) = x \otimes x$ and if $\varepsilon(x) = 1$. The set of group-like elements in C is denoted by $G(C)$.
- For any group-like elements $g, h \in G(C)$, the set of **(g, h)-primitive elements** of C is denoted by
$$P_{g,h}(C) := \{ x \in C \mid \Delta(x) = x \otimes g + h \otimes c \}.$$
- If B is a bialgebra, $P(B) := P_{1,1}(B)$. The elements of $P(B)$ are called primitive.

Example. For any Lie algebra $\mathfrak{g}$, and $x \in \mathfrak{g} \subseteq \mathcal{U}(\mathfrak{g})$ is primitive. i.e., $x \in P(\mathcal{U}(\mathfrak{g}))$.

LEM 4.6 If C is a coalgebra, $G(C)$ is a linearly independent subset of C.

Hint. Since ε is linear, $0 \notin G(C)$, so any single element of

$G(C)$ forms a linearly independent set.

Let $n := \min\{ m \in \mathbb{N} \mid$ any distinct m elements in $G(C)$ is linearly independent $\}$.

Then $n \geq 1$. Suppose $g = \lambda_1 g_1 + \cdots + \lambda_n g_n$ for distinct elements

$g, g_1, \cdots, g_n \in G(C)$, w/ $\lambda_1, \cdots, \lambda_n \in \mathbb{k}$. Then on the one hand,

$$\Delta(g) = g \otimes g = \sum_{i,j=1}^{n} \lambda_i \lambda_j \, g_i \otimes g_j$$

On the other hand, $\Delta(g) = \Delta\left(\sum_{j=1}^{n} \lambda_j g_j\right) = \sum_{j=1}^{n} \lambda_j \, g_j \otimes g_j$.

By assumption, $\{ g_i \otimes g_j \mid i, j = 1, \cdots, n \}$ is linearly independent,

so $n = 1$, $g = \lambda_1 g_1$. Apply ε on both sides, $1 = \lambda_1$, $\Rightarrow g = g_1$, $\rightarrow\leftarrow$!

$\boxtimes$

<u>Rmk</u>.

- For $C = \mathbb{k}G$, $G(C) = G$. However, for general coalgebra C', $G(C')$ may not have additional structures. For a bialgebra B, $G(B)$ is a monoid.

<u>Example</u> Let A be any algebra. Define $f: A \to \mathbb{k}$

$$Alg(A, \mathbb{k}) := \{ f \in A^* \mid f \text{ is an algebra map} \}.$$

$f \in Alg(A, \mathbb{k})$ iff $f(1_A) = 1$ ($\Uparrow f \in A^\circ$), and $\forall a, b \in A$,

$\langle \mu^*(f), a \otimes b \rangle = f(ab) = f(a) f(b) = \langle f \otimes f, a \otimes b \rangle$

iff $\mu^*(f) = f \otimes f$ and $\eta^*(f) = 1$ iff $f \in G(A^\circ)$.

i.e., $Alg(A, \mathbb{k}) = G(A^\circ)$.

Example. For any group G, consider $\Bbbk G$. By PROP 4.4, $R(G) := (\Bbbk G)^\circ$ is also a bialgebra, which is called the bialgebra of representative functions on G. By LEM 3.4,

$$R(G) = \{ f \in (\Bbbk G)^* \mid \dim(G \rightharpoonup f) < \infty \}$$

The multiplication on $R(G)$ is the pointwise multiplication. Indeed, $\forall f, g \in R(G)$ (or $(\Bbbk G)^*$) and $x \in G$,

$$(fg)(x) = \langle f \otimes g, \Delta(x) \rangle = \langle f \otimes g, x \otimes x \rangle = f(x)\, g(x).$$

Moreover, $\langle \mu^*(f), x \otimes y \rangle = f(xy)$, $\forall x, y \in G$.

In general, hard to know how to write $\mu^*(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}$ w/ explicit $f_{(1)}, f_{(2)}$. However, when G is finite, define, for any $x \in G$, $e_x \in (\Bbbk G)^*$ by $e_x(y) := \delta_{x,y}$, $\forall y \in G$.

So $\{ e_x \mid x \in G \}$ is a basis for $(\Bbbk G)^* = R(G)$. Then

$$\langle \mu^*(e_x), a \otimes b \rangle = \langle e_x, ab \rangle = \delta_{x,ab} = \sum_{yz=x} \langle e_y \otimes e_z, a \otimes b \rangle$$

$$\Rightarrow \mu^*(e_x) = \sum_{yz=x} e_y \otimes e_z.$$

Example. Let $B = \mathcal{O}(\underbrace{M_n(\Bbbk)}_{(n\times n)\text{-matrices over } \Bbbk}) = \Bbbk[X_{ij} \mid 1 \le i, j \le n]$, where

$$X_{ij} : M_n(\Bbbk) \to \Bbbk \qquad \text{"coordinate function"}$$
$$a \longmapsto a_{ij}$$

One can define a product on B by pointwise multiplication.

As an algebra, B is the comm. polynomial ring in the n^2 indeterminants $\{X_{ij}\}$.

One can also define a coproduct on B by "the dual of matrix multiplication".

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}.$$ By setting $\varepsilon(X_{ij}) = \delta_{i,j}$,

B becomes a bialgebra. If $X := [X_{ij}]$ the matrix of coordinate functions, then $\langle \det(X), a \rangle = \det(a)$ for $a \in M_n(\mathbb{k})$.

One can show that $\det(X) \in G(B)$. (by multiplicativity of det w.r.t. matrix multiplication).

<u>Rmk</u>. If B is a bialgebra, then $B \otimes B$, B^{op} (opposite product, same coproduct) B^{cop} (same product, opposite coproduct) and $B^{op, cop}$ are all bialgebras.

$*$: dual $\longrightarrow$ \ast, $*$

$\circledast$: conv. prod. $\longrightarrow$ \star

§5. Hopf algebra

<u>Def</u>. Let (A, μ_A, η_A) be an algebra and $(C, \Delta_C, \varepsilon_C)$ a coalgebra. Then $\mathrm{Hom}_{\mathbb{k}}(C, A)$ becomes an algebra under the convolution product

$$f * g := \mu_A \circ (f \otimes g) \circ \Delta_C$$

for all $f, g \in \mathrm{Hom}_{\mathbb{k}}(C, A)$ w/ unit element $\eta_A \circ \varepsilon_C : C \to \mathbb{k} \to A$.

In sigma notation: $$(f * g)(x) = \sum_{(x)} f(x_{(1)})\, g(x_{(2)})$$

<u>Example</u>. For any coalgebra C, the convolution product on $C^* = \mathrm{Hom}_{\mathbb{k}}(C, \mathbb{k})$ is exactly Δ^*.

<u>Def</u>. Let H be a bialgebra. An element $S \in \mathrm{Hom}_{\mathbb{k}}(H, H)$ which is an inverse to id_H under the convolution product is called an antipode of H.

<u>Rmk</u>. • Antipode may not exist.

- Unique if exists.
- In sigma notation, $S \in \mathrm{End}_{\mathbb{k}}(H)$ is an antipode of H if

$$\star \left[\begin{aligned} (S * id)(x) &= \sum_{(x)} S(x_{(1)})\, x_{(2)} = \eta\varepsilon(x) = \varepsilon(x)\cdot 1_H \\ &= (id * S)(x) = \sum_{(x)} x_{(1)}\, S(x_{(2)}), \quad \forall x \in H. \end{aligned} \right]$$

<u>DEF</u>. A Hopf algebra is 6-tuple $(H, \mu, \eta, \Delta, \varepsilon, S)$ s.t. $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, and $S \in \mathrm{End}_{\mathbb{k}}(H)$ is an antipode of H.

<u>Example</u>. Let G be a group. Extending $S(g) := g^{-1}$ $(\forall g \in G)$ linearly makes $\mathbb{k}G$ a Hopf algebra.

For any Hopf algebra H, and any $g \in G(H)$, we have

$$\sum_{(g)} S(g_{(1)})\, g_{(2)} = S(g)\cdot g = \varepsilon(g)\cdot 1_H = 1_H = g\cdot S(g) = \sum_{(g)} g_{(1)}\, S(g_{(2)})$$

$\Rightarrow$ g is invertible and $S(g) = g^{-1}$. Consequently, $G(H)$ is a group.