Lecture 4

Last time :

convolution product on $\mathrm{Hom}_{\mathbb{k}}(C, A)$

$$f * g := \mu_A \circ (f \otimes g) \circ \Delta_C$$

$$(f * g)(c) = \sum_{(c)} f(c_{(1)})\, g(c_{(2)}) \quad , \quad \forall c \in C$$

Antipode : for a bialgebra B, an antipode is an element $S \in \mathrm{Hom}_{\mathbb{k}}(B, B)$ that is inverse to id_H under $*$.

$$(S * id)(x) = \sum_{(x)} S(x_{(1)})\, x_{(2)} = \varepsilon(x) \cdot 1_H \quad (= \eta(\varepsilon(x)))$$

$$= \sum_{(x)} x_{(1)}\, S(x_{(2)}) = (id * S)(x)$$

Hopf algebra : a bialgebra w/ an antipode.

Recall : for any Hopf alg H, $\forall g, h \in \overset{\substack{\text{group like} \\ \uparrow}}{G}(H)$, $S(gh) = (gh)^{-1} = h^{-1} g^{-1}$
$= S(h) \cdot S(g)$.

<u>Prop 5.7</u> Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. Then

(1) S is an algebra anti-homomorphism

(2) S is a coalgebra anti-homomorphism , i.e., $\forall x \in H$,

$$(S \otimes S)\,\Delta(x) = \underset{\substack{\uparrow \\ \text{"swap"}}}{\tau}\; \Delta(S(x)) \quad \text{and} \quad \varepsilon(S(x)) = \varepsilon(x) .$$

<u>Sketch</u>. Consider the bialgebra $H \otimes H$ w/ counit $\varepsilon_{H \otimes H}(x \otimes y) = \varepsilon(x)\, \varepsilon(y)$.

Define $\varphi, \psi : H \otimes H \to H$ by

$\varphi(x\otimes y) := S(xy)$, $\quad \psi(x\otimes y) = S(y)S(x)$. $\quad (\forall x, y \in H)$

WTS: $\quad \varphi * \mu = \underbrace{\eta\circ\varepsilon_{H\otimes H}}_{\text{unit in } \mathrm{Hom}_{\mathbb{R}}(H\otimes H,\, H) \text{ under } *} = \mu * \psi \quad$ in $\mathrm{Hom}_{\mathbb{R}}(H\otimes H, H)$.

$\forall x, y \in H$,

$$(\varphi*\mu)(x\otimes y) = \mu\Big(\sum \varphi(x_{(1)}\otimes y_{(1)}) \otimes \mu(x_{(2)}\otimes y_{(2)})\Big)$$

$$= \sum S(x_{(1)}y_{(1)})\, x_{(2)}y_{(2)} = \sum S((xy)_{(1)})\,(xy)_{(2)} = \varepsilon(xy)\,1_H$$

$$= \varepsilon(x)\varepsilon(y)\,1_H = \eta\,\varepsilon_{H\otimes H}(x\otimes y).$$

$$(\mu*\psi)(x\otimes y) = \mu\Big(\sum \mu(x_{(1)}\otimes y_{(1)}) \otimes \psi(x_{(2)}\otimes y_{(2)})\Big)$$

$$= \sum x_{(1)}y_{(1)}S(y_{(2)})S(x_{(2)}) = \eta\,\varepsilon_{H\otimes H}(x\otimes y)$$

Therefore, $\quad \varphi = \varphi * (\mu * \psi) = (\varphi * \eta) * \psi = \psi$.

Moreover, $\quad 1_H = \varepsilon(1_H)\,1_H = S(1_H)\cdot 1_H = S(1_H)$. so S is an algebra anti-homomorphism.

To show S is a coalgebra anti-homomorphism, consider

$\Phi, \Psi : H \to H\otimes H$ by $\Phi := \Delta S$, $\quad \Psi := \tau(S\otimes S)\Delta$

Then show $\Phi * \Delta = \eta_{H\otimes H}\,\varepsilon = \Delta * \Psi$.

Hint: $(\Delta * \Psi)(x) = \mu_{H\otimes H}\Big(\sum \Delta(x_{(1)}) \otimes \Psi(x_{(2)})\Big)$

$$= \mu_{H\otimes H}\Big(\sum (x_{(1)}\otimes x_{(2)}) \otimes (S(x_{(4)}) \otimes S(x_{(3)}))\Big)$$

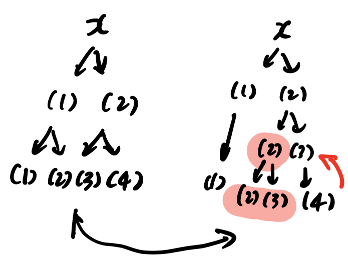
$$= \sum x_{(1)}S(x_{(4)}) \otimes x_{(2)}S(x_{(3)})$$

$$= \sum x_{(1)}S(x_{(3)}) \otimes \varepsilon(x_{(2)})\,1_H$$

$$= \sum x_{(1)}S(\varepsilon(x_{(2)})\,x_{(3)}) \otimes 1_H$$

$$= \sum x_{(1)}S(x_{(2)}) \otimes 1_H$$

$$= \varepsilon(x)\,1_H \otimes 1_H.$$



$\blacksquare$

In sigma notation, $\sum S(x_{(1)}) \otimes S(x_{(2)}) = \sum (Sx)_{(2)} \otimes (Sx)_{(1)}$

As a consequence, $S^2 = S \circ S$, is an algebra homomorphism.

<u>Prop 5.9</u> TFAE.

(1) $\sum S(x_{(2)})\, x_{(1)} = \varepsilon(x) 1_H = \eta\varepsilon(x), \quad \forall x \in H.$

(2) $\sum x_{(2)}\, S(x_{(1)}) = \varepsilon(x) 1_H = \eta\varepsilon(x), \quad \forall x \in H$

(3) $S^2 = id.$

Consequently, if H is commutative or cocommutative, then $S^2 = id$.

<u>Pf</u>. (1) $\Rightarrow$ (3). Assume (1)

$$(S * S^2)(x) = \sum S(x_{(1)})\, S^2(x_{(2)}) \overset{\text{Prop 5.7}}{=} S\left(\sum S(x_{(2)})\, x_{(1)}\right) \overset{(1)}{=} S(\varepsilon(x) 1_H)$$

$$= \varepsilon(x) 1_H = \eta\varepsilon(x).$$

so $S^2 = \eta\varepsilon * S^2 = id * (S * S^2) = id$.

(3) $\Rightarrow$ (1). Assume (3).

$$\sum S(x_{(2)})\, x_{(1)} = \sum S(x_{(2)})\, S^2(x_{(1)}) = S\left(\sum S(x_{(1)})\, x_{(2)}\right) = S(\varepsilon(x) 1_H)$$

$$= \varepsilon(x) 1_H = \eta\varepsilon(x).$$

(2) $\Leftrightarrow$ (3) is proved similarly. ∎

<u>Example</u>. $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra w/ antipode $S(x) = -x$. $\forall x \in \mathfrak{g}$.
cocommutative.

<u>Example</u>. Taft algebras.

Let $n \geq 2$ be an integer and assume $\mathbb{k}$ contains a primitive n-th root of unity ζ.

The Taft algebra $T_{n^2}(\xi)$ is a $\Bbbk$-algebra defined by

$$T_{n^2}(\xi) := \Bbbk \langle x, g \mid x^n = 0,\ g^n = 1,\ xg = \xi g x \rangle$$

clearly, $\dim_{\Bbbk}(T_{n^2}(\xi)) = n^2$ w/ a basis $\{ g^i x^j \mid i, j = 0, \cdots, n-1 \}$.

$T_{n^2}(\xi)$ has the following coalgebra structure and antipode:

$$\Delta(g) = g \otimes g, \qquad \Delta(x) = x \otimes 1 + g \otimes x,$$

$$\varepsilon(g) = 1, \qquad \varepsilon(x) = 0$$

$$S(g) = g^{-1}, \qquad S(x) = -g^{-1}x.$$

$T_{n^2}(\xi)$ is non-commutative, non-cocommutative Hopf algebra.

$\downarrow$

if $\mathrm{ord}(S) < \infty$, then S has to have even order.

$$S^2(g) = g, \quad S^2(x) = S(S(x)) = S(-g^{-1}x) = -\underbrace{S(x)}\underbrace{S(g^{-1})}$$

$$= g^{-1} x g = \xi x$$

$\Rightarrow \quad S^2(g^i x^j) = \xi^j g^i x^j. \quad \forall\ 0 \le i, j \le n-1.$

$\Rightarrow \quad \mathrm{ord}(S^2) = n$, or $\mathrm{ord}(S) = 2n$.

The smallest non-comm., non-co comm. Hopf alg is $T_4(-1)$.

Rmk. $\exists$ Hopf algebra whose antipode has infinite order. (Sweedler's book).

By integral theory, such Hopf alg must be ∞-dim'l.

Prop 5.11. If $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra, then $(H^\circ, \Delta^*, \varepsilon^*, \mu^*, \eta^*, S^*)$ is also a Hopf algebra.

<u>Sketch</u>. By <u>Prop 4.4</u>, it remains to show that $S^*(H^\circ) \subseteq H^\circ$, and S^* satisfies the antipode conditions.

$\forall\, a, b \in H,\ f \in H^*,\quad \langle a \rightharpoonup S^*(f), b\rangle = \langle S^*(f), ba\rangle = \langle f, S(ba)\rangle$
$= \langle f, S(a)S(b)\rangle = \langle f \leftharpoonup S(a), S(b)\rangle = \langle S^*\big(f \leftharpoonup S(a)\big), b\rangle$

Therefore, $H \rightharpoonup S^*(f) = S^*(f \leftharpoonup S(H)) \subseteq S^*(f \leftharpoonup H)$. Hence, if $f \in H^\circ$, then by Lem 3.4, $S^*(f) \in H^\circ$.

To check the antipode condition of S^*: $\forall\, x \in H,\ f \in H^\circ$,

$$\sum_{(f)} \langle S^*(f_{(1)}) f_{(2)}, x\rangle = \sum_{(f)} \langle \Delta^*(S^*(f_{(1)}) \otimes f_{(2)}), x\rangle$$

$$= \sum_{(f),(x)} \langle S^*(f_{(1)}), x_{(1)}\rangle \langle f_{(2)}, x_{(2)}\rangle = \sum_{(f),(x)} \langle f_{(1)}, S(x_{(1)})\rangle \langle f_{(2)}, x_{(2)}\rangle$$

$$= \sum_{(f),(x)} \langle \underbrace{f_{(1)} \otimes f_{(2)}}_{\mu^*(f)}, S(x_{(1)}) \otimes x_{(2)}\rangle$$

$$= \sum_{(x)} \langle f, S(x_{(1)}) x_{(2)}\rangle = \varepsilon(x) f(1_H) = \langle \varepsilon^* \eta^*(f), x\rangle.$$

The other conditions are left as exercise. ▨

§6. Modules and comodules

Recall: a left module of a $\mathbb{k}$-algebra A is a pair (M, γ) where M is a $\mathbb{k}$-space, $\gamma: A \otimes M \to M$ is a $\mathbb{k}$-linear map such that the following diagrams commute

$$\begin{tikzcd} A\otimes A\otimes M \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\gamma"'] & A\otimes M \arrow[d, "\gamma"] \\ A\otimes M \arrow[r, "\gamma"] & M \end{tikzcd} \qquad \begin{tikzcd} \Bbbk\otimes M \arrow[r, "\eta\otimes id"] \arrow[dr, "\cong"'] & A\otimes M \arrow[d, "\gamma"] \\ & M \end{tikzcd}$$

The category of left A-modules is denoted by ${}_A Mod$.

Mod_A

<u>Example</u>. Let A be an algebra. Recall we have defined $a \rightharpoonup f \in A^*$ by $\langle a \rightharpoonup f, b\rangle = \langle f, ba\rangle$, $\forall a, b \in A$, $f \in A^*$. Easy to check: $(A^*, \rightharpoonup) \in {}_A Mod$.

<u>DEF</u>. For a coalgebra C, a right C-comodule is a pair (M, ρ) where M is a $\Bbbk$-space, $\rho: M \to M\otimes C$ is a linear map s.t.

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M\otimes C \arrow[d, "id\otimes\Delta"] \\ M\otimes C \arrow[r, "\rho\otimes id"] & M\otimes C\otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "\otimes 1"'] & M\otimes C \arrow[d, "id\otimes\varepsilon"] \\ & M\otimes\Bbbk \end{tikzcd}$$

are commutative. The category of right C-comodules is denoted by $Cmod^C$.

${}^{C}Cmod$

Sigma notation for right comodules: $\rho(m) = \sum m_{(0)} \otimes m_{(1)} \in M\otimes C$, we understand that $m_{(i)} \in C$ for all $i \neq 0$. Similary, for a left C-comodule $(N, \rho': N \to C\otimes N)$, write

$$\rho'(n) = \sum n_{(-1)} \otimes n_{(0)} \in C\otimes N.$$

A comodule homomorphism b/w (M, ρ_M) and $(N, \rho_N) \in \mathrm{Cmod}^C$ is a linear map $f : M \to N$ s.t.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M"'] & N \arrow[d, "\rho_N"] \\ M \otimes C \arrow[r, "f \otimes id"] & N \otimes C \end{tikzcd}$$

is commutative.

RMK 6.3. Recall: if C is a coalgebra, then C^* is an algebra.
Suppose $(M, \rho) \in \mathrm{Cmod}^C$ so that $\forall\, m \in M$, $\rho(m) = \sum_i m_{(0)} \otimes m_{(1)}$.
Define $\tilde{\rho} : C^* \otimes M \to M$ by $\tilde{\rho}(f \otimes m) := \sum_i f(m_{(1)})\; m_{(0)} \in M$ for any $f \in C^*$, then one can easily check $(M, \tilde{\rho}) \in {}_{C^*}\mathrm{Mod}$.
The converse is not true in general. For an arbitrary $(M, \gamma) \in {}_{C^*}\mathrm{Mod}$, there may not be $\rho : M \to M \otimes C$ s.t. $(M, \rho) \in \mathrm{Cmod}^C$ and $\tilde{\rho} = \gamma$.
A C^*-module which has a C-comodule structure in the above fashion is called rational.

Example. C is a C-comodule (left and right) via $\rho : C \to C \otimes C$.

LEM 6.4 Suppose A is an algebra. If (M, ρ) is a right A°-comodule, then w/ the natural A-action $a \cdot m = \sum_{i=1}^{r} f_i(a)\, m_i$ (if $\rho(m) = \sum_{i=1}^{r} m_i \otimes f_i$)
not sigma notation
$A \cdot m$ is finite-dim'l for all $m \in M$.
Conversely, if M is a left A-module, and $A \cdot m$ is finite-dim'l for all $m \in M$, then M is naturally a right A°-comodule.

Sketch. (1) $\forall\, m \in M$ w/ $\rho(m) = \sum_{i=1}^{r} m_i \otimes f_i \in M \otimes A^\circ$, we have

$A \cdot m \subseteq \mathrm{span}_{\mathbb{k}} \{ m_1, \dots, m_r \}$. (2) Conversely, fix $m \in M$, and let $\{m_1, \cdots, m_r\}$ be a basis for $A\cdot m$. Then for any $a \in A$, $a\cdot m = \sum_{i=1}^{r} f_i(a)\, m_i$ for some $f_i \in A^*$. Now all the functionals f_i vanishes on the ideal $\ker(A \xrightarrow{\text{mult. in } A} \mathrm{End}_{\mathbb{k}}(A\cdot m))$ which is cofinite, so $f_i \in A^\circ$. We are done by defining $\rho: M \to M \otimes A^\circ$, $\rho(m) = \sum_{i=1}^{r} m_i \otimes f_i$. $\blacksquare$

<u>Example</u>. Consider $(C, \Delta) \in \mathrm{Comod}^C$. By the above, we have a C^*-module structure on C by

$$f \rightharpoonup c := \sum f(c_{(2)})\, c_{(1)} \in C, \quad \forall f \in C^*,\ c \in C.$$

Alternatively, $\forall f, g \in C^*$, $\langle g, f \rightharpoonup c\rangle = \sum \langle f, c_{(2)}\rangle\langle g, c_{(1)}\rangle = \langle gf, c\rangle$.

"$\rightharpoonup$ is the transpose of right mult. of C^*."

$(C, \rightharpoonup) \in {}_{C^*}\mathrm{Mod}$.

Similarly, $(C, \leftharpoonup) \in \mathrm{Mod}_{C^*}$ w/ $\langle g, c \leftharpoonup f\rangle = \langle fg, c\rangle$ or

$$c \leftharpoonup f = \sum f(c_{(1)})\, c_{(2)}. \quad \forall f, g \in C^*,\ c \in C.$$