Lecture 5

Last time: comodule C (coalg)

$\rho : M \to M \otimes C$ coassociativity

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\wr"] & M \otimes C \arrow[d, "id \otimes \Delta"] \\ M \otimes C \arrow[r, "\rho \otimes id"'] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[dr, "\otimes 1"'] & M \otimes C \arrow[d, "id \otimes \varepsilon"] \\ & M \otimes \Bbbk \end{tikzcd}$$

$$\rho(m) = \sum m_{(0)} \otimes m_{(1)} = \sum m_0 \otimes m_1 .$$

- A C-comodule is naturally a C^*-module.

C is a comodule of itself via $\Delta : C \to C \otimes C$

$\Downarrow$

C is a C^*-module: $\forall f \in C^*$, $\forall x \in C$,

$$f \rightharpoonup c := \sum f(c_2)\, c_1 \in C$$

$$c \leftharpoonup f := \sum f(c_1)\, c_2 \in C$$

A right subcomodule D of C is also known as a right coideal of C. that means $\Delta(D) \subseteq D \otimes C$. Similarly, a left coideal E of C is a left subcomodule, i.e., $\Delta(E) \subseteq C \otimes E$.

Rmk. Every left C^*-submodule $D \subseteq C$ is a right coideal. (This is a special case when the converse of the above holds). Choose $d \in D$. Write $\Delta(d) = \sum_{i=1}^{n} d_i \otimes d_i'$ for some $d_1, \dots, d_m \in D$ and $d_{m+1}, \dots, d_n$

not sigma notation!

are linearly independent mod D. For any $f \in C^*$, $f \rightharpoonup d = \sum_{i=1}^{n} f(d_i') \cdot d_i \in D$
By construction, $f(d_i') = 0$ for all $i > m$. Since f is arbitrary, $d_i' = 0$ for all $i > m$, i.e., $\Delta(d) \in D \otimes C$.

Prop 6.6. Let G be any group. A vector space M has a $\Bbbk G$-comodule structure if and only if it is a G-graded module, that is, $M = \bigoplus_{g \in G} M_g$.

Sketch. Assume (M, ρ) is a right $\Bbbk G$-comodule. Write $\rho(m) = \sum_{g \in G} m_g \otimes g$

for any $m \in M$. Coassociativity $\Rightarrow \sum_{g,h \in G} (m_g)_h \otimes h \otimes g = \sum_{g \in G} m_g \otimes g \otimes g$

so $\rho(m_g) = m_g \otimes g$. Setting $M_g := \{ m_g \mid m \in M \}$, then $M_g \cap M_h = \delta_{g,h} M_g$. Moreover, the counit condition implies $\forall m \in M$

$$(id \otimes \varepsilon)\Big(\sum_{g \in G} m_g \otimes g\Big) = \sum_{g \in G} m_g = m \otimes \underset{1 \in \Bbbk}{\underset{\uparrow}{1}} = m. \quad \text{So } M = \bigoplus_{g \in G} M_g.$$

Conversely, if $M = \bigoplus_{g \in G} M_g$, then one can check directly that

$\rho : M \to M \otimes \Bbbk G$, sending $m = \sum_{g \in G} m_g$ to $\rho(m) := \sum_{g \in G} m_g \otimes g$ is a

right $\Bbbk G$-comodule structure on M. $\boxtimes$

Now assume H is a Hopf algebra.

Def. Let M be a left H-module. The invariants of H in M are elements of the set $M^H := \{ m \in H \mid h \cdot m = \varepsilon(h) m, \ \forall h \in H \}$.

Let (M, ρ) be a right H-comodule. The coinvariants of H in M are elements of the set

$$M^{coH} := \{ m \in M \mid \rho(m) = m \otimes 1_H \}$$

Example.

- A trivial left H-module is an H-module M s.t. $M^H = M$.
- Let $H = \Bbbk G$. If M is a left H-module, then $M^H = M^G$. If M is a right H-comodule, then $M^{coH} = M_e$, in light of the decomposition $M = \bigoplus_{g \in G} M_g$ (PROP 6.6).

A direct consequence of LEM 6.4 (last time) implies

PROP 6.9. (1) Let (M, ρ) is a right H-comodule, and consider its left H^*-module structure, then $M^{H^*} = M^{coH}$.

(2) Let M be a left H-module s.t. it is also a right H°-comodule. Then $M^H = M^{coH^\circ}$. □

The whole next chapter will focus on H^H, where H acts on itself by left / right multiplication.

Tensor product of modules / comodules.

Given $(V, \varphi_V), (W, \varphi_W) \in {}_H\mathrm{Mod}$, $V \otimes W$ is naturally an H-module via Δ:
$$h \cdot (v \otimes w) := \sum_{(h)} (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot w)$$
$\forall h \in H, v \in V, w \in W$.

In terms of maps, the module structure on $V \otimes W$ is
$$\varphi_{V \otimes W} := (\varphi_V \otimes \varphi_W) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes id \otimes id)$$
$$H \otimes V \otimes W \to H \otimes H \otimes V \otimes W \to H \otimes V \otimes H \otimes W \to V \otimes W.$$

Similarly, if $(V, \rho_V), (W, \rho_W) \in \mathrm{Cmod}^H$, then $(V \otimes W, \rho_{V\otimes W})$ is also a right H-comodule, where

$$\rho_{V\otimes W} := (id \otimes id \otimes \mu) \circ (id \otimes \tau \otimes id) \circ (\rho_V \otimes \rho_W)$$

$$V \otimes W \to V \otimes H \otimes W \otimes H \to V \otimes W \otimes H \otimes H \to V \otimes W \otimes H.$$

i.e., $\rho_{V\otimes W}(v \otimes w) = \sum v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)}$.

<u>DEF</u>. Let H be a Hopf algebra. A **right H-Hopf module** is a triple $(M, \gamma : M \otimes H \to M, \rho : M \to M \otimes H)$ s.t.

(1) (M, γ) is a right H-module

(2) (M, ρ) is a right H-comodule

(3) $\rho : M \to M \otimes H$ is a right H-module map, where H acts on itself by right multiplication.

Can rewrite the last condition in the above def as

"
$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "(\,)\cdot h"'] & M \otimes H \arrow[d, "(\,)\cdot h"] \\ M \arrow[r, "\rho"] & M \otimes H \end{tikzcd}$$
"

$$\sum m_0 \cdot h_1 \otimes m_1 h_2 \quad \text{"}\rightarrow\downarrow\text{"}$$
$$= \sum (m \cdot h)_0 \otimes (m \cdot h)_1 \quad \text{"}\downarrow\rightarrow\text{"} \qquad (*)$$

We can also replace condition (1) w/ the condition that M is a right K-module, where K is a Hopf subalgebra "sub Hopfalgebra" of H, then M is called a right (H, K)-Hopf module. The category of all right (H, K)-Hopf modules is denoted by $H\mathrm{mod}_K^H$. It is easy to define ${}^H H\mathrm{mod}_K$, ${}_K H\mathrm{mod}^H$ and ${}_K^H H\mathrm{mod}$.

<u>Example</u>. • H itself is an H-Hopf module via $\rho = \Delta$.

- For any $W \in \mathrm{Mod}_H$, $W \otimes H$, equipped w/ the natural module structure, is an H-Hopf module via $id_W \otimes \Delta : W \otimes H \to W \otimes H \otimes H$.

- Let W be a trivial right H-module, consider the right H-Hopf module $W \otimes H$ as above. For all $w \in W$, and $a, b \in H$, we have $(w \otimes a) \cdot b = \sum_1 w \cdot b_1 \otimes a b_2 = \sum_1 w \cdot \varepsilon(b_1) \otimes a b_2$ $= w \otimes ab$. In other words, the H-module structure on $W \otimes H$ is $id_W \otimes \mu$

For any vector space V, it is easy to check that $(V \otimes H, id_V \otimes \mu, id_V \otimes H)$ is a right H-Hopf module. We call it the trivial right H-Hopf module structure on $V \otimes H$.

THM 6.12 (Fundamental Theorem of Hopf Modules)

Let H be a Hopf algebra and $M \in {}_H\mathrm{mod}_H^H$. Then $M \cong M^{coH} \otimes H$ as right H-Hopf modules, where $M^{coH} \otimes H$ is endowed w/ the trivial right H-Hopf module structure. In particular, M is a free right H-module of rank $\dim_k (M^{coH})$.

Sketch. Define $\beta : M \to M \otimes H$ by $m \mapsto \sum_1 m_0 \cdot S(m_1) \otimes m_2$.

We first show that $\beta(M) \subseteq M^{coH} \otimes H$. by showing

$$\rho(\textstyle\sum_1 m_0 \cdot S(m_1)) = \sum_1 m_0 \cdot S(m_1) \otimes 1_H. \text{ for all } m \in M.$$

$$\rho(\textstyle\sum_1 m_0 \cdot S(m_1)) = \sum_1 (m_0)_0 \cdot (S(m_1))_1 \otimes (m_0)_1 \cdot (S(m_1))_2$$

(by def of Hopf modules (*))

$$= \textstyle\sum_1 m_{0,0} \cdot S(m_{1,2}) \otimes m_{0,1} \cdot S(m_{1,1}) \quad \text{(anti-comultiplicativity of } S\text{)}$$

$= \sum m_0 \cdot S(m_3) \otimes m_1 \cdot S(m_2) = \sum m_0 \cdot S(m_2) \otimes \varepsilon(m_1) \cdot 1_H$

$= \sum m_0 \cdot S(m_1) \otimes 1_H .$

Define $\alpha : M^{coH} \otimes H \to M$ by $m' \otimes h \mapsto m' \cdot h$ $\forall m' \in M^{coH},\ h \in H$, we check α and β are inverse to each other. Indeed, $\forall m' \in M^{coH},\ h \in H$,

$$\beta\alpha(m' \otimes h) = \beta(m' \cdot h) = \sum (m' \cdot h)_0 \cdot S((m' \cdot h)_1) \otimes (m' \cdot h)_2$$

Since $m' \in M^{coH}$ and ρ is a Hopf module map, we have

$$(\rho \otimes id)(\rho(m' \cdot h)) = \sum m' \cdot h_1 \otimes h_2 \otimes h_3$$

$\uparrow \rho \otimes id = \uparrow id \otimes \Delta$. (coassociativity)

$$\rho(m') \cdot h \underset{\text{module map}}{=} (m' \otimes 1) \cdot h = m' \cdot h_1 \otimes h_2$$

So $\beta\alpha(m' \otimes h) = \sum (m' \cdot h_1) \cdot S(h_2) \otimes h_3 = \sum m' \cdot (h_1 S(h_2)) \otimes h_3$

$= \sum m' \varepsilon(h_1) \otimes h_2 = m' \otimes h .$

Conversely, for any $m \in M$,

$$\alpha\beta(m) = \alpha\left(\sum m_0 \cdot S(m_1) \otimes m_2\right) = \sum m_0 \cdot S(m_1) m_2 = \sum m_0 \otimes \varepsilon(m_1)$$

$\overset{\text{counit}}{=} m .$

i.e.,
$$\begin{tikzcd} M^{coH} \otimes H \arrow[r, "\alpha"] \arrow[d, "id \otimes \Delta"'] & M \arrow[d, "\rho"] \\ M^{coH} \otimes H \otimes H \arrow[r, "\alpha \otimes id"] & M \otimes H \end{tikzcd}$$

Now we check α is a right H-comodule map. Choose any $m' \in M^{coH}$, $h, k \in H$, we have

$$\rho\alpha(m' \otimes h) = \rho(m' \cdot h) \overset{m' \in M^{coH}}{=} \sum m' \cdot h_1 \otimes h_2$$

$= (\alpha \otimes id)(id \otimes \Delta)(m' \otimes h)$ ✓

Finally, we check α is an H-module map.

$$\begin{tikzcd} M^{coH} \otimes H \arrow[r, "\alpha"] \arrow[d, "\cdot k"'] & M \arrow[d, "\cdot k"] \\ M^{coH} \otimes H \arrow[r, "\alpha"] & M \end{tikzcd}$$

$(\alpha(m' \otimes h)) \cdot k = (m' \cdot h) \cdot k = m' \cdot (hk)$

$= \alpha((m' \otimes h) \cdot k) = \alpha(m' \otimes hk)$

$\uparrow id \otimes \mu$

So α is an isomorphism of right H-Hopf-modules. $\boxtimes$

Example. Let G be any group, $H = \Bbbk G$, and let M be any right $\Bbbk G$-Hopf module.

$\Rightarrow$ (Prop 6.6) $M = \bigoplus_{g \in G} M_g$, and $\rho(m_g) = m_g \otimes g$. for any $m_g \in M_g$.

Also, G acts on M and ρ being a right H-module map means $\rho(m \cdot h) = \rho(m) \cdot h$. for all $h \in G$. That is, $\rho(m_g \cdot h) = m_g \cdot h \otimes gh$

$\Rightarrow$ $M_g \cdot h = M_{gh}$ for all $g, h \in G$. In particular, $M_e \cdot g = M_g$.

We can compare this w/ the Fundamental Theorem : here, $M^{coH} = M_e$, and $M \cong M_e \otimes \Bbbk G$ as $\Bbbk G$-Hopf modules implies that $M_g \cong M_e \otimes g$.

Although by Fundamental Theorem, all Hopf-modules are trivial, but in practice, the difficulty is to prove a vector space has a H-Hopf module structure, so that the Fundamental Theorem can be applied. We will see this in Chapter 2.