Lecture 6

Last time: Fundamental Theorem of Hopf modules $M \cong M^{coH} \otimes H$

$M^{coH} = \{ m \in M \mid \rho(m) = m \otimes 1_H \}$

Chapter 2 Integral theory.

In this chapter, all Hopf algebras are assumed to be finite-dim'l, unless otherwise stated.

§2.1 Definition and existence

DEF. Let H be a finite-dim'l Hopf algebra. A left integral in H is an element $\Lambda^L \in H$ such that $h\Lambda^L = \varepsilon(h)\Lambda^L$ for all $h \in H$;
a right integral in H is an element $\Lambda^R \in H$ such that $\Lambda^R h = \varepsilon(h)\Lambda^R$ for all $h \in H$. The space of left (resp. right) integrals in H is denoted by $\int_H^L$ (resp. $\int_H^R$), and H is called unimodular if $\int_H^L = \int_H^R$.

Example. Note that H^* is also a Hopf algebra. A left integral in H^* is also called a left cointegral of H. By definition, such an element $\lambda^L \in H^*$ satisfies $\varphi \cdot \lambda^L = \varepsilon_{H^*}(\varphi)\lambda^L$ for all $\varphi \in H^*$. Equivalently, $\lambda^L \in H^*$ is a cointegral of H if and only if for any $\varphi \in H^*$ and $x \in H$,

$$\langle \varphi \cdot \lambda^L, x \rangle = \sum \varphi(x_1)\lambda^L(x_2) = \varphi\Big(\sum x_1 \cdot \lambda^L(x_2)\Big)$$

def of cointegral $\rightsquigarrow$ $\parallel$

$$\varphi(1_H \cdot \lambda^L(x)),$$

def of $\cdot$ in H^* + linearity

if and only if $(id \otimes \lambda^L)\Delta(x) = \lambda^L(x) \cdot 1_H$.

Similarly, a right cointegral of H is an element $\lambda^R \in H^*$ s.t.

$(\lambda^R \otimes id)\,\Delta(x) = \lambda^R(x)\cdot 1_H$ for all $x \in H$.

Example.

- If $H = \Bbbk G$ for a finite group G, then $\Lambda = \sum_{g\in G} g$ is both left and right integral in H, and $\int_H^L = \int_H^R = \Bbbk\Lambda$.

  In $H^* = (\Bbbk G)^*$, $\lambda = \delta_e \in H^*$ is a left and right cointegral of H, for instance, $(id \otimes \delta_e)\,\Delta(g) = (id \otimes \delta_e)(g \otimes g) = \delta_e(g)\, g = \delta_e(g) e$ for all $g \in G$. Moreover, $\int_{H^*}^L = \int_{H^*}^R = \Bbbk\lambda$. Both H and H^* are unimodular.

- If H is commutative, and it has left and right integrals, then H is unimodular.

  Basis : $1, x, g, gx$, $x^2 = 0$, $g^2 = 1$, $xg = -gx$

- The Sweedler (Taft) algebra $H = T_4(-1)$ is not unimodular.

  It is easy to check that $\Lambda^L = x + gx$ is a left integral:

  $x\Lambda^L = x^2 + xgx = 0 - gx^2 = 0 = \varepsilon(x)\Lambda^L$

  $g\Lambda^L = gx + g^2x = x + gx = \Lambda^L = \varepsilon(g)\Lambda^L$.

  In addition, $\int_H^L = \Bbbk\Lambda^L$. Similarly, $\Lambda^R = x - gx$ is a right integral, and $\int_H^R = \Bbbk\Lambda^R$.

- If H is cocommutative, and has left and right integrals, it may not be unimodular : assume $\mathrm{char}\,\Bbbk = 2$. and $\mathfrak{g}$ be the 2-dim'l Lie algebra $\mathfrak{g} = \Bbbk\langle x, y \mid [x,y] = x\rangle$. Endow $\mathcal{U}(\mathfrak{g})$ with the usual Hopf algebra structure, and let $B \subseteq \mathcal{U}(\mathfrak{g})$ be the ideal generated by x^2 and $y^2 - y$, then B is a Hopf ideal and $H = \mathcal{U}(\mathfrak{g})/B$ is a cocommutative 4-dim'l Hopf algebra with basis $\{\bar{1}, \bar{x}, \bar{y}, \overline{xy}\}$. One can check that $\int_H^L = \Bbbk\,\bar{x}\bar{y}$ while $\int_H^R = \Bbbk\,\overline{yx} = \Bbbk(\overline{xy} + \bar{x})$.

A finite-dim'l $\Bbbk$-algebra A is a Frobenius algebra if there exists a non-degenerate associative bilinear form $(\cdot,\cdot) : A \otimes A \to \Bbbk$, where associativity means $(a, bc) = (ab, c)$ for all $a, b, c \in A$.

For example, $M_n(\Bbbk)$ is a Frobenius algebra with $(a,b) := Tr(ab)$.

<u>THM 1.4</u> (Larson-Sweedler)

Let H be any finite-dimensional Hopf algebra. Then

(1) $\dim_{\Bbbk}(\int_H^L) = \dim_{\Bbbk}(\int_H^R) = 1$

(2) the antipode S of H is bijective, and $S(\int_H^L) = \int_H^R$.

(3) H is a cyclic left and right H^*-module.

(4) H is a Frobenius algebra.

To prove the theorem, we need the following observations.

1. H^* is a left H^*-module via left multiplication

$$\Downarrow$$

H^* is a right H-comodule by LEM 6.4 (Chap 1).

More precisely, if $\{\varphi_1, \dots, \varphi_n\}$ is a basis of H^* and let $f \in H^*$ be an arbitrary element, then there exists $h_1^f, \dots, h_n \in H$ s.t. for any $g \in H^*$, $gf = \sum_{i=1}^{n} \langle g, h_i \rangle \varphi_i$.

The H-comodule structure on H^* is then

$$\rho : H^* \longrightarrow H^* \otimes H, \quad \rho(f) := \sum_{i=1}^{n} \varphi_i \otimes h_i$$

(*)

Conversely, if $\rho(f) = \sum f_0 \otimes f_1$, then $\boxed{gf = \sum \langle g, f_1 \rangle f_0}$.

2. H^* is a right H-module via $\langle f \leftharpoonup h, \ell \rangle := \langle f, \ell \cdot S(h) \rangle$ for any $f \in H^*$, $h, \ell \in H$. Equivalently, $f \leftharpoonup h = S(h) \rightharpoonup f$

LEM 1.5. $(H^*, \leftharpoonup, \rho) \in \mathrm{Hmod}_H^H$.

PF. To show that ρ is a right H-module homomorphism.

$$\begin{tikzcd} H^* \arrow[r, "\rho"] \arrow[d, "\leftharpoonup h"'] \arrow[dr, phantom, "\circlearrowright"] & H^* \otimes H \arrow[d, "\cdot h"] \\ H^* \arrow[r, "\rho"] & H^* \otimes H \end{tikzcd} \quad \Leftrightarrow \quad \rho(\underbrace{f \leftharpoonup h}_{\psi}) = \rho(f) \cdot h = \sum (\overbrace{f_0 \leftharpoonup h_1}^{\psi_0}) \otimes \overbrace{f_1 h_2}^{\psi_1}$$

for all $f \in H^*$ and $h \in H$.

As noted above, the information of $\rho(f \leftharpoonup h)$ can extracted from $g \cdot (f \leftharpoonup h)$ when g varies in H^*. so it suffices to show that

$$g(f \leftharpoonup h) = \sum \langle g, f_1 h_2 \rangle (f_0 \leftharpoonup h_1)$$

First, we study $g(f \leftharpoonup h)$. For any $x \in H$, on the one hand,

$$\langle g(f \leftharpoonup h), x \rangle = \langle g(Sh \rightharpoonup f), x \rangle = \sum \langle g, x_1 \rangle \langle Sh \rightharpoonup f, x_2 \rangle$$
$$= \sum \langle g, x_1 \rangle \langle f, x_2 S(h) \rangle.$$

On the other hand,

$$\sum \langle (h_2 \rightharpoonup g) f \leftharpoonup h_1, x \rangle = \sum \langle (h_2 \rightharpoonup g) f, x S(h_1) \rangle$$
$$= \sum \langle h_2 \rightharpoonup g, (x S(h_1))_1 \rangle \langle f, (x S(h_1))_2 \rangle$$

$\hookrightarrow x_1 (S(h_1))_1 = \underbrace{x_1 S(h_{1,2})}_{\text{"S"}}$ $\hookrightarrow x_2 (S(h_1))_2 = \underbrace{x_2 S(h_{1,1})}_{\text{"S"}}$

$$= \sum \langle h_3 \rightharpoonup g, x_1 S(h_2) \rangle \langle f, x_2 S(h_1) \rangle$$
$$= \sum \langle g, x_1 S(h_2) h_3 \rangle \langle f, x_2 S(h_1) \rangle$$
$$= \sum \langle g, x_1 \rangle \langle f, x_2 S(h) \rangle$$

"$\left(\underbrace{S(h_2) h_3}_{\downarrow} \otimes h_1 \right.$
$\left. \varepsilon(h_2) 1_H \otimes h_1 \rightsquigarrow 1_H \otimes h \right)$"

So $\boxed{g(f \leftharpoonup h) = \sum (h_2 \rightharpoonup g) f \leftharpoonup h_1}$. (**)

Therefore, by (*) above and (**)

$$g(f \leftharpoonup h) = \sum \underbrace{(h_2 \rightharpoonup g) f}_{\text{apply } (*)} \leftharpoonup h_1 = \sum \big(\langle h_2 \rightharpoonup \overset{\overset{H}{\downarrow}}{g}\,, f_1 \rangle f_0\big) \leftharpoonup h_1$$

$$= \sum \langle h_2 \rightharpoonup g, f_1 \rangle (f_0 \leftharpoonup h_1) = \sum \langle g, f_1 h_2 \rangle (f_0 \leftharpoonup h_1)$$

as desired. ▨

<u>PF of THM 1.4</u>.

By LEM 1.5, $M := H^* \in {}_H\mathrm{mod}_H^H$, so by the Fundamental Theorem of Hopf modules, $M \cong M^{coH} \otimes H$. Since $\dim(M) = \dim(H^*) = \dim(H)$, so $\dim(M^{coH}) = 1$. By PROP 6.9 (Chap. 1), $M^{coH} = M^{H^*} = (H^*)^{H^*}$

By the above discussions, the right H-comodule structure on H^* corresponds to the left H^*-module structure (given by left multiplication), so

$$M^{coH} = (H^*)^{H^*} = \{ f \in H^* \mid \varphi f = \varepsilon_{H^*}(\varphi) f \} = \int_{H^*}^{L},$$

and so $\dim\left(\int_{H^*}^{L}\right) = 1$ for any finite-dim'l Hopf algebra H. Applying this to H^*, $\dim\left(\int_{H}^{L}\right) = 1$.

Now choose $0 \neq \lambda \in \int_{H^*}^{L}$, then $\int_{H^*}^{L} = \Bbbk\lambda$. Let

$$\alpha : \Bbbk\lambda \otimes H = M^{coH} \otimes H \longrightarrow M, \quad \alpha(\lambda \otimes h) = \lambda \leftharpoonup h = S(h) \rightharpoonup \lambda$$

be the map in the proof of the Fundamental Theorem of Hopf modules, then for any $k \in \ker(S)$, $\alpha(\lambda \otimes k) = 0$. Since α is injective, $\lambda \neq 0$, so $k = 0$, which means S is injective. By dimension counting, S is also surjective, so S is bijective. Consequently, $S\left(\int_H^L\right) = \int_H^R$

because $S(\Lambda) \cdot h = S(S^{-1}h \cdot \Lambda) = S(\varepsilon(S^{-1}h) \cdot \Lambda) = \varepsilon(h) S(\Lambda) \Rightarrow S\Lambda \in \int_H^R$ for any $\Lambda \in \int_H^L$. Thus, $\dim\left(\int_H^R\right) = 1$. We have (1) and (2).

Let $\lambda \neq 0$ be as above. We have $H^* \overset{\text{Fund. THM}}{=} \lambda \leftharpoondown H \overset{\text{def}}{=} SH \rightharpoonup \lambda \overset{(2)}{=} H \rightharpoonup \lambda$

Dualizing the equalities gives (3). Finally, define a bilinear form

$$(\cdot,\cdot) : H \otimes H \longrightarrow \Bbbk, \quad (h,k) := \langle \lambda, hk \rangle, \ \forall h,k \in H.$$

It is easy to see that the form is bilinear and associative, and it remains to show $(\cdot,\cdot)$ is non-degenerate. Since H is finite-dimensional, we only need to show left non-degeneracy. Assume there exists $h \in H$ s.t. for any $k \in H$, $0 = (h,k) = \langle \lambda, hk \rangle = \langle k \rightharpoonup \lambda, h \rangle$, so $\langle H \rightharpoonup \lambda, h \rangle = 0$. By (3), $H \rightharpoonup \lambda = H^*$, so $\langle H^*, h \rangle = 0$. This means $h = 0$, and we are done. ▨

<u>Remark</u>. If A is a Frobenius algebra in the above sense, then $A \cong A^*$ via the form $(\cdot,\cdot)$. Then one can define a coalgebra structure by $(\Delta(x), a \otimes b) = (x, ab)$ and $\varepsilon(x) = (x, 1_A)$ for all $x, a, b \in A$.

The algebra and coalgebra structure can be generalized to define Frobenius algebra objects in tensor categories. (related to topological quantum field theory).

<u>Example</u>. When $H = \Bbbk G$ for a finite group G, the bilinear form is determined by $\lambda = \delta_e \in \int^L_{H^*}$, thus for $x = \sum_{g \in G} a_g g$, $y = \sum_{h \in G} b_h h$,

$$(x,y) = \langle \delta_e, xy \rangle = \sum_{g \in G} a_g b_{g^{-1}} \in \Bbbk.$$