Lecture 7

Integral theory (for finite-dimensional Hopf alg H)

- $h\Lambda^l = \varepsilon(h)\Lambda^l$, $\forall h \in H$.
- THM 1.4. $\forall$ Hopf alg H of $\dim < \infty$, $\dim \int_H^l = \dim \int_H^R = 1$, S is bijective, $S(\int_H^l) = \int_H^R$, H is a Frobenius algebra.

§2.2. Maschke's Theorem

Let R be a ring, and M is a non-zero R-module. We call M reducible if there is a proper submodule $0 \neq N \subsetneq M$, otherwise M is irreducible. If M is a direct sum of irreducible R-modules, then M is called completely reducible. In particular, the ring R is semisimple if R is completely reducible as an R-module.

[Curtis - Reiner] Representation theory of finite group and associative algebras.

The classical Maschke's Theorem says that if G is a finite group, then $\Bbbk G$ is semisimple if and only if $|G|$ is invertible in $\Bbbk$.

Translating this into the language of integrals, let $\Lambda := \sum_{g\in G} g \in \int_{\Bbbk G}$, then

units in $\Bbbk$

$\varepsilon(\Lambda) = |G|$, and $|G| \in \Bbbk^{\times}$ if and only if $\varepsilon(\Lambda) \neq 0$.

THM 2.1 (LS) Let H be a finite-dimensional Hopf algebra. Then H is semisimple if and only if $\varepsilon(\int_H^l) \neq 0$, if and only if $\varepsilon(\int_H^R) \neq 0$.

PF. Assume H is semisimple. Since $\ker(\varepsilon)$ is an ideal of H, by complete reducibility, we may write $H = I \oplus \ker(\varepsilon)$ for some left ideal $I \neq 0$ of H. We claim $I \subseteq \int_H^L$. Choose $z \in I$, and $h \in H$. Since $h - \varepsilon(h)1_H \in \ker(\varepsilon)$, so $hz = \underbrace{(h - \varepsilon(h)1_H)z}_{=0} + \varepsilon(h)z = \varepsilon(h)z$, and so $z \in \int_H^L$. Since I is one-dimensional, $I = \int_H^L$, and we may choose $0 \neq \Lambda \in I$ s.t. $\varepsilon(\Lambda) \neq 0$ as $\Lambda \notin \ker(\varepsilon)$. Thus, $\varepsilon(\int_H^L) \neq 0$. Similarly, $\varepsilon(\int_H^R) \neq 0$.

Conversely, assume $\varepsilon(\int_H^L) \neq 0$. Then we choose $\Lambda \in \int_H^L$ s.t. $\varepsilon(\Lambda) = 1$. Let M be any left H-module, and N an H-submodule. We will show that N has a complement in H. Let $\pi : M \to N$ be any k-linear projection. and define

$$\tilde{\pi} : M \to N, \quad \tilde{\pi}(m) := \sum \Lambda_1 \cdot \pi(S(\Lambda_2)\cdot m), \quad \forall m \in M.$$

We claim that $\tilde{\pi}$ is an H-projection of M onto N.

Firstly, for any $n \in N \subseteq M$, $\tilde{\pi}(n) = \sum \Lambda_1 \cdot (S(\Lambda_2)\cdot n) = \sum (\Lambda_1 S(\Lambda_2))\cdot n = \varepsilon(\Lambda)\cdot n = n$. so $\tilde{\pi}$ is a linear projection. To see $\tilde{\pi}$ is an H-map, note that for any $h \in H$,

$$\sum \Lambda_1 \otimes \Lambda_2 \otimes h = \Delta(\Lambda) \otimes h = \Delta(\Lambda) \otimes \left(\sum \varepsilon(h_1) h_2\right)$$
$$= \sum \Delta(\varepsilon(h_1)\Lambda) \otimes h_2 = \sum \Delta(h_1 \Lambda) \otimes h_2 = \sum h_1\Lambda_1 \otimes h_2\Lambda_2 \otimes h_3$$

Therefore, for any $h \in H$,

$$\tilde{\pi}(h \cdot m) = \sum \Lambda_1 \cdot \pi(S(\Lambda_2)\cdot h \cdot m)$$
$$= \sum h_1\Lambda_1 \cdot \pi(S(h_2\Lambda_2)h_3 \cdot m)$$
$$= \sum \underbrace{h_1}_{h}\Lambda_1 \cdot \pi(S(\Lambda_2)\underbrace{S(h_2)h_3}_{\varepsilon(h_2)\cdot 1_H} \cdot m)$$
$$= \sum h\Lambda_1 \cdot \pi(S(\Lambda_2)\cdot m) = h \cdot \sum \Lambda_1 \cdot \pi(S(\Lambda_2)\cdot m) = h \cdot \tilde{\pi}(m).$$

Hence, $\ker(\tilde{\pi})$ is an H-complement for N, so M is completely reducible. Therefore, H is semisimple. Finally, assume $\varepsilon(\int_H^R) \neq 0$, we can apply similar argument to right modules. $\boxtimes$

<u>Rmk.</u> THM 2.1 does not require the use of THM 1.4 : the existence of integrals follows from semisimplicity of H in this case.

A $\Bbbk$-algebra A is separable (over $\Bbbk$) if $\overset{A \otimes_{\Bbbk} E}{\overset{\parallel}{A \otimes E}}$ is a semisimple algebra over E for every extension field $E/\Bbbk$. In particular, separable algebras are necessarily semisimple. An intrinsic characterization of separability is given as follows:

a $\Bbbk$-algebra A is separable if and only if the multiplication $\mu : A \otimes A \to A$ admits a section $\sigma : A \to A \otimes A$ s.t. σ is an A-A bimodule map, i.e., write $\sigma(1_A) = \sum_i a_i \otimes b_i \in A \otimes A$, then

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a\cdot"', "\cdot a"] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "a\cdot", "\cdot a"'] \\ A \arrow[r] & A \otimes A \end{tikzcd}$$

$$\sigma(a) = \sum_i a a_i \otimes b_i = \sum_i a_i \otimes b_i a, \quad \text{and} \quad \sum_i a_i b_i = 1_A.$$

In this case, the section σ is completely determined by the element $p := \sigma(1_A) \in A \otimes A$, which is called the separability idempotent associated to σ. (By def. $p^2 = p$). See [CR].

<u>Example</u>. Fix $n \in \mathbb{N}$. Let $e_{ij} \in M_n(\Bbbk)$ be the (i,j)-th matrix unit, then $\{ e_{ij} \mid i,j = 1, \cdots, n \}$ forms a basis for $M_n(\Bbbk)$. Fix an integer $1 \leq x \leq n$, and define $p_x := \sum_{j=1}^{n} e_{jx} \otimes e_{xj} \in M_n(\Bbbk) \otimes M_n(\Bbbk)$, then p_x is a separability idempotent of $M_n(\Bbbk)$. Indeed, $\mu(p_x) = \sum_{j=1}^{n} e_{jx} e_{xj} = \sum_{j=1}^{n} e_{jj} = id_n = 1_{M_n(\Bbbk)}$

and for any $1 \leq a, b \leq n$,

$$e_{ab} \cdot p_x = \sum_j e_{ab}\, e_{jx} \otimes e_{xj} = e_{ax} \otimes e_{xb} = \sum_j e_{jx} \otimes e_{xj}\, e_{ab} = p_x \cdot e_{ab}.$$

Therefore, $M_n(\Bbbk)$ is separable.

<u>Cor 2.4</u> Let H be a finite-dimensional semisimple Hopf algebra over $\Bbbk$. Then H is separable over $\Bbbk$, and for any Hopf subalgebra $K \subseteq H$ such that H is free over K, we have K semisimple.

<u>Sketch</u>. Let $E/\Bbbk$ be any field extension. We will show that $\underbrace{H \otimes E}_{H \otimes_{\Bbbk} E}$ is semisimple. Note that $H \otimes E$ is a Hopf algebra over E via

$$\Delta(h \otimes \alpha) := \Delta(h) \otimes \alpha \in H \otimes_{\Bbbk} H \otimes_{\Bbbk} E \cong (H \otimes_{\Bbbk} E) \otimes_E (H \otimes_{\Bbbk} E).$$

$$\varepsilon(h \otimes \alpha) := \varepsilon(h) \cdot \alpha \in E.$$

$$S(h \otimes \alpha) := S(h) \otimes \alpha \in H \otimes_{\Bbbk} E.$$

for all $h \in H$, $\alpha \in E$. It follows that $\int_H^L \otimes E = \int_{H \otimes E}^L$. So by THM 2.1, $H \otimes E$ is semisimple.

Now let K be a Hopf subalgebra of H s.t. H is free over K, and let $\{h_i\}$ be a free basis of H as a K-module. Choose $\Lambda \in \int_H^L$ w/ $\varepsilon(\Lambda) \neq 0$, we may write $\Lambda = \sum_i k_i h_i$ for some $k_i \in K$. Then for any $x \in K$,

$$\sum_i (x k_i)\, h_i = x\Lambda = \varepsilon(x)\Lambda = \sum_i (\varepsilon(x) k_i)\, h_i$$

Since $\{h_i\}$ is a free K-basis, we have $x k_i = \varepsilon(x) k_i$ for all i, i.e., $k_i \in \int_K^L$ for all i. Now $\varepsilon(\Lambda) = \sum \varepsilon(k_i)\,\varepsilon(h_i) \neq 0$ implies that for some j, $\varepsilon(k_j) \neq 0$, so K is semisimple by THM 2.1. $\blacksquare$

<u>Rmk</u>. In fact, all finite-dimensional Hopf algebras are free over Hopf subalgebras, and so all Hopf subalgebras of a finite-dimensional semisimple Hopf algebra are semisimple.

Let H be a finite-dim'l Hopf algebra. For any $0 \neq \Lambda^L \in \int_H^L$ $\curvearrowright$ $\int_H^L = \Bbbk \Lambda^L$, we have $\Lambda^L h \in \int_H^L$. Since $\dim(\int_H^L) = 1$, so $\Lambda^L h = \alpha(h) \Lambda^L$, where $\alpha(h) \in \Bbbk$.

It is clear that $\alpha \in H^*$, and for any $h, k \in H$,

$$\alpha(hk)\Lambda^L = \Lambda^L(hk) = (\Lambda^L h)k = \alpha(h)\Lambda^L k = \alpha(h)\alpha(k)\Lambda^L, \text{ so}$$

$$\alpha(hk) = \alpha(h)\alpha(k), \text{ i.e., } \alpha \in Alg(H, \Bbbk) = \underbrace{G(H^*)}_{\text{group like elements in } H^*}$$

DEF. Let H be a finite-dimensional Hopf algebra. The distinguished grouplike element of H^* is the functional $\alpha \in G(H^*)$ such that $\Lambda^L h = \alpha(h)\Lambda^L$ for all $\Lambda^L \in \int_H^L$ and $h \in H$. The distinguished grouplike element of H is the element $g \in G(H)$ such that $(\lambda^L \otimes id)\Delta(h) = \lambda^L(h) g$ for all $\lambda^L \in \int_{H^*}^L$ and $h \in H$.

If we start w/ $0 \neq \Lambda^R \in \int_H^R$, then $S(\Lambda^R) \in \int_H^L$, so for any $h \in H$,

$$h\Lambda^R = S^{-1}(S(\Lambda^R)S(h)) = \underbrace{\alpha(S(h))}_{\alpha \in G(H^*)} \cdot S^{-1}(S(\Lambda^R)) = \underbrace{\alpha^{-1}}_{\text{inverse in the algebra } H^*}(h)\Lambda^R.$$

By def, H is unimodular if and only if $\alpha = \varepsilon$.

COR. If H is semisimple, then H is unimodular.

PF. By THM 2.1, we may choose $0 \neq \Lambda \in \int_H^L$ w/ $\varepsilon(\Lambda) \neq 0$. Let $\alpha \in G(H^*)$ be the distinguished grouplike element, then for any $h \in H$,

$$\alpha(h)\varepsilon(\Lambda)\Lambda = \underline{\alpha(h)\Lambda} \cdot \Lambda = \Lambda \underline{h \Lambda} = \varepsilon(h)\Lambda^2 = \varepsilon(h)\varepsilon(\Lambda)\Lambda$$

Since $\varepsilon(\Lambda) \neq 0$, we must have $\alpha(h) = \varepsilon(h)$ for all $h \in H$. i.e., H is unimodular.

§2.3. Commutative semisimple Hopf algebras.

An easy example of such a Hopf algebra is $H = (\Bbbk G)^*$ for a finite group G.

Up to extension of field, this is the only example.

THM 3.1 (Cartier) Let H be a finite-dim'l commutative semisimple Hopf algebra over $\Bbbk$, then there exists a group G and a separable extension field $E/\Bbbk$ such that $H \otimes_{\Bbbk} E \cong (EG)^*$ as Hopf algebras.

Sketch. Since H is commutative and semisimple, by Artin-Wedderburn, $H = \bigoplus_i E_i$, where E_i are fields containing $\Bbbk$. Moreover, each E_i is separable over $\Bbbk$ by COR 2.4. Thus, we may choose E to be a common extension field of all the E_i, which is separable $/\Bbbk$. It follows that $H \otimes E \cong E^{\oplus n}$, where $n = \dim(H)$.
WLOG, we may assume that $H \cong \Bbbk^{\oplus n}$.
Let $\{p_i \mid i=1, \cdots, n\}$ be a basis of orthogonal idempotents of H, and $G := \{g_i \mid i=1, \cdots, n\}$ be the corresponding dual basis for H^*. Since each $g_i \in Alg(H, \Bbbk) = G(H^*)$, so $G \subseteq G(H^*)$. Moreover, by LEM 4.6 (Chap 1) $G(H^*)$ is linearly independent, so $G(H^*) = G$ is a finite group, and $H^* = \Bbbk G$ is the group algebra. $\boxtimes$