Lecture 8

Last time : Maschke's Theorem (§2.2)

If H is a finite-dim'l Hopf algebra, then H is semisimple if and only if $\varepsilon(\int_H^L) \neq 0$, if and only if $\varepsilon(\int_H^R) \neq 0$.

Recall in the proof of "$\Leftarrow$", to find the orthogonal complement of $N \subseteq M$ (H-mod),
we used $\tilde{\pi}(m) := \sum \Lambda_1 \cdot \pi(S(\Lambda_2) \cdot m)$ for $m \in M$ and $\pi : M \to N$ linear projection, $\Lambda \in \int_H^L$.

$\hookrightarrow$ closely related to the Frobenius algebra structure of H.

- distinguished grouplike element
- commutative semisimple Hopf algebra. $H \otimes_{\Bbbk} E \cong (EG)^*$, $E/\Bbbk$ separable, G = finite group.

$\hookrightarrow$ §2.3.

Continue §2.3 w/ a family of Lie algebras over positive characteristic introduced by Jacobson.

<u>DEF</u>. Let $\Bbbk$ be a field of characteristic $p \neq 0$. A restricted Lie algebra over $\Bbbk$ is a pair $(\mathfrak{g}, (\cdot)^{[p]})$, where $(\cdot)^{[p]} : \mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^{[p]}$ is a map s.t. for all $a \in \Bbbk$, $x, y \in \mathfrak{g}$,

- $(ax)^{[p]} = a^p x^{[p]}$
- $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i}$, where $s_i(x,y)$ is the coefficient of t^{i-1} in $\underset{\substack{\| \\ [tx+y, \cdots [tx+y, [tx+y, x]]]}}{ad_{tx+y}^{p-1}(x)}$, and $ad_v(w) := [v,w]$ for $v, w \in \mathfrak{g}$.
- $ad_{x^{[p]}} = ad_x^p$

If A is any algebra over $\Bbbk$ w/ $\operatorname{char}(\Bbbk) = p \neq 0$, then the associated Lie algebra A_L is restricted w/ $[a, b] := ab - ba$ and $a^{[p]} = a^p$ for all $a, b \in A$.
(see the note of Jared Warner on restricted Lie algebras)

Let $(\mathfrak{g}, (\cdot)^{[p]})$ be a restricted Lie algebra, and let $U(\mathfrak{g})$ be the usual enveloping algebra. Let B be the ideal in $U(\mathfrak{g})$ generated by elements of the form $x^p - x^{[p]}$ for all $x \in \mathfrak{g}$, and define $u(\mathfrak{g}) = U(\mathfrak{g})/B$, then $u(\mathfrak{g})$ is called the restricted enveloping algebra, or u-algebra, of $\mathfrak{g}$.
In fact, $u(\mathfrak{g})$ can be defined using the universal property of $U(\mathfrak{g})$.
As for $U(\mathfrak{g})$, $\mathfrak{g}$ embeds into $u(\mathfrak{g})_L$ via $x \mapsto x + B$, and the $[p]$-map is the usual p-power map under this embedding.

A version of PBW theorem holds for $u(\mathfrak{g})$: given a basis for $\mathfrak{g}$, the ordered monomials in this basis, where the exponent of each element is bounded by $p-1$, form a basis for $u(\mathfrak{g})$.

$$
\boxed{
\begin{array}{ll}
e_{i_1}^{a_1} e_{i_2}^{a_2} \cdots \in U(\mathfrak{g}) & \\[2ex]
e_{i_1}^{a_1} e_{i_2}^{p} \cdots + B & \in u(\mathfrak{g}) \\
= e_{i_1}^{a_1} \underbrace{e_{i_2}^{[p]}}_{\in \mathfrak{g}} \cdots + B & \\
\quad \| & \\
e_{i_1}^{a_1} \left(\sum_k^{\ell} e_{j_k} b_k \right) \cdots + B &
\end{array}
}
$$

Consequently, if $\mathfrak{g}$ has $\dim n < \infty$, then $\dim(u(\mathfrak{g})) = p^n$.

Finally, $u(\mathfrak{g})$ inherits the Hopf algebra structure from $U(\mathfrak{g})$, as one can check B is actually a Hopf ideal.

THM 3.3 (Hochschild)

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra over $\mathbb{k}$ of characteristic $p \neq 0$. Then $u(\mathfrak{g})$ is semisimple if and only if $\mathfrak{g}$ is abelian and is spanned over $\mathbb{k}$ by $\mathfrak{g}^{[p]}$.

Pf. Let E be the algebraic closure of $\mathbb{k}$. Since $u(\mathfrak{g} \otimes E) = u(\mathfrak{g}) \otimes E$ is semisimple if and only if $u(\mathfrak{g})$ is semisimple, and $(\mathfrak{g} \otimes E)^{[p]}$ spans $\mathfrak{g} \otimes E$ over E if and only if $\mathfrak{g}$ is spanned by $\mathfrak{g}^{[p]}$ over $\mathbb{k}$. So we may assume WLOG that $\mathbb{k} = \overline{\mathbb{k}}$ (alg. closed).

Assume $\mathfrak{g}$ is abelian and is spanned by $\mathfrak{g}^{[p]}$. Then $H = u(\mathfrak{g})$ is a commutative Hopf algebra and the p-map is semilinear. Moreover, $H = H^p$, so $(\cdot)^p : H \to H$ is injective, and so H has no non-zero nilpotent elements, so it is semisimple.

Conversely, assume $u(\mathfrak{g})$ is semisimple. We first show that for any $x \in \mathfrak{g}$, $x \in \langle x \rangle^p \subseteq u(\mathfrak{g})$, where $\langle x \rangle := \mathrm{span}_{\mathbb{k}} \{ x^{[p]^i} \mid i \in \mathbb{N} \} \subseteq \mathfrak{g}$. By definition, $\langle x \rangle$ is an abelian restricted Lie subalgebra of $\mathfrak{g}$. By the restricted PBW, $H = u(\mathfrak{g})$ is free over $K = u(\langle x \rangle)$, so by Corollary 2.4, K is semisimple. Therefore, there exists $\Lambda \in \int_K^l$ w/ $\varepsilon(\Lambda) \neq 0$.

If $\dim(\langle x \rangle) = n$, then by definition, x satisfies a polynomial of the form
$$f(x) = \sum_{i=0}^{n} a_i x^{p^i} = 0 \quad \text{w/ } a_i \in \mathbb{k},\ a_n \neq 0. \quad (\text{WTS: } a_0 \neq 0).$$
identify $x^{[p]}$ w/ x^p in $u(\mathfrak{g})$
$$\| $$
$$a_0 x + a_1 x^p + \cdots + a_n x^{p^n}$$

(In fact, by the restricted PBW, $f(x)$ is the minimal polynomial of x in K).
Moreover, $\Lambda \in K$ can be written uniquely as $\Lambda = g(x) = \sum_{j=0}^{p^n-1} b_j x^j$ w/ $b_j \in \mathbb{k}$, and $b_0 = \varepsilon(\Lambda) \neq 0$. By definition, $x\Lambda = \varepsilon(x)\Lambda = 0$, so

$f(x)$ divides $x g(x)$. Comparing degrees, we have $d f(x) = x g(x)$ for some $d \in \Bbbk^\times$, i.e., $x(b_0 + b_1 x + \cdots) = d(a_0 x + a_1 x^p + \cdots)$. Since $b_0\, d \neq 0$, we have

$a_0 \neq 0$, and so $x = \sum_{i=1}^{n} c_i x^{p^i}$, where $c_i = -\frac{a_i}{a_0} \in \Bbbk = \Bbbk^p$, $c_n = -\frac{a_n}{a_0} \neq 0$, so

$x \in \langle x \rangle^p$, and this proves $\mathfrak{g}$ is spanned by $\mathfrak{g}^{[p]}$.

Finally, x satisfies a separable polynomial. Consequently, ad_x satisfies a separable polynomial, and its action on $\mathfrak{g}$ is completely reducible. Let $y \in \mathfrak{g}$ be an eigenvector of ad_x. Then ad_x acts on the commutative ring $u(\langle y \rangle)$, and by definition, ad_x annihilates $\langle y \rangle^p$.

$$
\begin{aligned}
ad_x(y^p) &= [x, y^p] = -[y^p, x] = -ad_{y^p}(x) \\
&= -ad_{y^{[p]}}(x) = -ad_y^p(x) = -[y, \cdots, [y, \underbrace{[y, x]}] \cdots] \\
&= [y, \cdots, [y, \underset{\substack{\| \\ ad_x(y) = \lambda y}}{[x, y]}] \cdots] = 0.
\end{aligned}
$$

However, $y \in \langle y \rangle^p$ by the above, so $ad_x(y) = 0$. Since $\mathfrak{g}$ is spanned by eigenvectors of ad_x, we have $ad_x(\mathfrak{g}) = 0$, so x is central. Since x is arbitrary, $\mathfrak{g}$ is abelian. ∎

LEM 3.4 Let $\mathfrak{g}$ be a Lie algebra (resp., restricted Lie algebra) over characteristic $p \neq 0$. If $f \in U(\mathfrak{g})^\circ$ (resp., $u(\mathfrak{g})^\circ$) is an algebra homomorphism, then $f^p = \varepsilon$.

Sketch. Let $H = U(\mathfrak{g})$, it suffices to show that $f^p(1_H) = 1_H$ and $f^p(x) = 0$, $\forall x \in \mathfrak{g}$. By def, $f(1_H) = 1_H$, so $f^n(1_H) = f^{\otimes n}(\Delta^{[n]}(1_H)) = (f(1_H))^n = 1$ for all $n \geq 0$.

For $x \in \mathfrak{g}$, we use induction to show $f^n(x) = n f(x)$. When $n=1$, trivial.
When $n \geq 2$, by induction hypothesis, $f^n(x) = (f * f^{n-1})(x)$
$= (f \otimes f^{n-1})(x \otimes 1_H + 1_H \otimes x) = f(x) \cdot f^{n-1}(1_H) + f(1_H) f^{n-1}(x)$
$= f(x) + (n-1) f(x) = n f(x)$.
In particular, $f^p(x) = p f(x) = 0$. The restricted case follows similarly.

$\blacksquare$

<u>COR 3.5</u> Let $\mathrm{char}(\Bbbk) = p \neq 0$, and $\mathfrak{g}$ a restricted Lie algebra over $\Bbbk$ of dimension $n < \infty$ such that $u(\mathfrak{g})$ is semisimple. Then for some finite separable field extension $E/\Bbbk$, we have $u(\mathfrak{g}) \otimes E \cong (E\Gamma)^*$, where $\Gamma \cong (\mathbb{Z}/p\mathbb{Z})^n$.

<u>PF</u>. By THM 3.3, $H = u(\mathfrak{g})$ is commutative. By THM 3.1, $u(\mathfrak{g}) \otimes E \cong (E\Gamma)^*$ where $E/\Bbbk$ is separable and $\Gamma = G(H^*) = \mathrm{Alg}(H, \Bbbk)$. By definition, $u(\mathfrak{g})$ is cocommutative, so Γ is an abelian group. Moreover, $|\Gamma| = \dim(u(\mathfrak{g})) = p^n$, and by LEM 3.4, $\mathrm{ord}(g) = p$ for any nontrivial element $g \in \Gamma$. Therefore, $\Gamma \cong (\mathbb{Z}/p\mathbb{Z})^n$.

$\blacksquare$.

<u>RMK</u>. There is also a dual notion of semisimplicity for coalgebras. Namely, one can define simple coalgebras and simple / completely reducible comodules of a coalgebra in the same way as one did for algebras and modules. One can show that a coalgebra is a direct sum of simple coalgebras if and only if every comodule is completely reducible.

"cosemisimple"

If C is a finite-dim'l coalgebra, then C is a cosemisimple coalgebra

$\Updownarrow$

C^* is a semisimple algebra.

For example, $\Bbbk G$ is cosemisimple for any group G. since each $g \in G$ generates a 1-dim'l subcoalgebra $\Bbbk g \subseteq \Bbbk G$, which is simple.
When G is finite, $(\Bbbk G)^* \cong \Bbbk^{\oplus n}$ is certainly semisimple.

For a Hopf algebra H of arbitrary dimension, one can generalize the notion of a left/right cointegral in H^* (note that H^* itself may not be a Hopf algebra as $\dim(H)$ can be ∞): such an element $\lambda \in H^*$ satisfies $f\lambda = f(1_H)\lambda$ for all $f \in H^*$. For example, let G be a compact topological group, and $H = R^c(G)$ be the Hopf algebra of continuous complex-valued representative functions on G. Consider the Haar measure τ on G, and define $\lambda \in H^*$ by $\lambda(h) := \int_G h(x)\, d\tau(x)$. The translation invariance of τ and Peter-Weyl imply that λ is a cointegral in the above sense.

The dual Maschke's Theorem states that an arbitrary Hopf algebra is cosemisimple if and only if $\exists\, \lambda \in H^*$ cointegral s.t. $\lambda(1_H) = 1$. More structures of comodules and coalgebras will be discussed in the next chapter.

Next time: two big thms on s.s. Hopf alg,
order of antipode, trace function,
Need to review: Frobenius algebra (Curtis-Reiner
Methods of rep theory I,
§9)

distinguished grouplike elements.