Lecture 9

Last time : commutative Hopf algebra , restricted Lie algebra

§2.4. Semisimplicity and order of antipode.

Goal : explain two theorems of Larson-Radford (conjectured by Kaplansky)

Assume $\mathrm{char}(\mathbb{k}) = 0$, H is a finite-dim'l Hopf algebra / $\mathbb{k}$. In this case, H^* is semisimple if and only if H is cosemisimple. We will sketch the proof of :

<u>THM 4.1</u> $S^2 = id_H$ if and only if H is semisimple and cosemisimple.
(H^* is semisimple)

<u>THM 4.2</u> H is semisimple if and only if H^* is semisimple.

We present a simplified version of the proof due to Schneider, who used properties of Frobenius algebras in an essential way. So we first recall some basic facts on Frobenius algebras w/out proof.

Let A be any finite-dim'l $\mathbb{k}$-algebra, then

- A^* is a left A-module via

  $A \otimes A^* \to A^*$, $\quad a \otimes f \mapsto a \rightharpoonup f$, $\quad (a \rightharpoonup f)(b) = f(ba)$

- A^* is a right A-module via

  $A^* \otimes A \to A^*$, $\quad f \otimes a \mapsto f \leftharpoonup a$, $\quad (f \leftharpoonup a)(b) = f(ab)$

for all $f \in A^*$, $a, b \in A$.

The proof of the following lemma can be found in [Curtis - Reiner, Methods of Representation Theory Vol. I, Sec. 9A, 9B].

LEM 4.3 Let A be a finite-dim'l algebra of dimension n. TFAE.

- A is a Frobenius algebra.
- $\exists f \in A^*$ s.t. the map $\Phi : A \to A^*$, $\Phi(a) := (a \rightharpoonup f)$, is a left A-module isomorphism.
- $\exists f \in A^*$ s.t. the map $\Psi : A \to A^*$, $\Psi(a) := (f \leftharpoonup a)$, is a right A-module isomorphism.
- $\exists f \in A^*$ and $r_i, l_i \in A$, $i = 1, \dots, n$, s.t. for any $a \in A$, we have $a = \sum_{i=1}^{n} r_i f(l_i a) = \sum_{i=1}^{n} f(a r_i) l_i$. $\square$

The tuple (f, r_i, l_i) is called a Frobenius system for A. Given such a system, an associative non-deg. bilinear form on A is given by $(a, b) := f(ab)$. Moreover, the elements r_i, l_i form a dual basis w.r.t. this form, i.e., $(l_i, r_j) = \delta_{ij}$. Such a dual basis is not unique, but the element $\sum_{i=1}^{n} r_i \otimes l_i$ is uniquely determined by the bilinear form.

Now let H be a finite-dim'l Hopf algebra $/\Bbbk$. Recall the definition of the distinguished grouplike elements in H and H^*:

- $\alpha \in G(H^*)$ s.t. $\Lambda^L h = \alpha(h) \Lambda^L$, $\forall\, 0 \neq \Lambda^L \in \int_H^L$, $h \in H$.
- $g \in G(H)$ s.t. $(\lambda^L \otimes id)\Delta(h) = \lambda^L(h) g$ $\forall\, 0 \neq \lambda^L \in \int_{H^*}^L$, $h \in H$.

LEM 4.4. Choose $0 \neq \lambda^L \in \int_{H^*}^L$.

(1) Let $\Lambda \in H$ be such that $\lambda^L \leftharpoonup \Lambda = \varepsilon$, then $\Lambda \in \int_H^R$ and $\lambda^L(\Lambda) = 1$.

(2) Let $g \in H$ be the distinguished grouplike element, then $g \rightharpoonup \lambda^L \in \int_{H^*}^R$, and similarly, $\lambda^L \leftharpoonup g \in \int_{H^*}^R$. Moreover, for any $t \in \int_H^R$, we have $(g \rightharpoonup \lambda^L)(t) = \lambda(t)$.

<u>Pf</u>. (1) By THM 1.4, H is a Frobenius algebra w/ the bilinear form given by λ^L. By LEM 4.3, $A \to A^*$, $x \mapsto (\lambda^L \leftharpoonup x)$ is an A-module isom. Since for any $h \in H$, $\lambda^L \leftharpoonup (\Lambda h) = (\lambda^L \leftharpoonup \Lambda) \leftharpoonup h = \varepsilon \leftharpoonup h = \varepsilon(h)\,\varepsilon = \lambda^L \leftharpoonup (\varepsilon(h)\Lambda)$, so $\Lambda \in \int_H^R$. Moreover, $\lambda^L(\Lambda) = (\lambda \leftharpoonup \Lambda)(1_H) = \varepsilon(1_H) = 1$.

(2) By def, for any $f \in H^*$, $f\lambda^L = \underbrace{f(1_H)}_{\varepsilon^*(f)}\lambda^L$, and $\lambda^L f = f(g)\,\lambda^L$. Now

g is grouplike, so for any $\varphi, \psi \in H^*$, we have

$$\big(g \rightharpoonup (\varphi\psi)\big)(h) = (\varphi\psi)(hg) = \sum \varphi\big((hg)_1\big)\cdot\psi\big((hg)_2\big)$$
$$= \sum \varphi(h_1 g)\,\psi(h_2 g) = \sum (g \rightharpoonup \varphi)(h_1)\cdot(g \rightharpoonup \psi)(h_2) = \big[(g \rightharpoonup \varphi)\cdot(g \rightharpoonup \psi)\big](h).$$

for any $h \in H$. Therefore, for any $h \in H$, we have

$$\langle (g \rightharpoonup \lambda^L) f, h \rangle = \langle g \rightharpoonup [\lambda^L (g^{-1} \rightharpoonup f)], h \rangle = \langle \lambda^L (g^{-1} \rightharpoonup f), hg \rangle$$
$$= \langle g^{-1} \rightharpoonup f, g \rangle \cdot \lambda^L(hg) = f(1_H)\cdot\lambda^L(hg) = \underbrace{f(1_H)}_{\boxed{\varepsilon^*(f)}}\langle g \rightharpoonup \lambda^L, h \rangle.$$

So $g \rightharpoonup \lambda^L \in \int_{H^*}^R$, and $\lambda^L \leftharpoonup g \in \int_{H^*}^R$ can be similarly proved. Finally, if $t \in \int_H^R$, then $(g \rightharpoonup \lambda^L)(t) = \lambda^L(tg) = \lambda^L(\varepsilon(g)t) = \lambda^L(t)$. ∎

<u>DEF</u>. Let A be a finite dim'l Frobenius algebra w/ non-deg associative bilinear form $(\cdot,\cdot)$. The Nakayama automorphism of A is the map $N : A \to A$ determined by $(a, b) = (b, N(a))$ for all $a, b \in A$.

Note that given a Frobenius system (f, r_i, l_i) of a Frobenius algebra, then the Nakayama automorphism depends only on f.

Recall that for any coalgebra C, C^* acts on C from left and right via

$$f \rightharpoonup c := \sum f(c_2)\, c_1 \quad , \quad c \leftharpoonup f = \sum f(c_1)\, c_2$$

For a finite-dim'l Hopf algebra, this is precisely the action of H^* on $H = H^{**}$ mentioned above. We state the following results of Schneider w/out proof, although it is the key technical result we need to prove THM 4.1, 4.2.

(See Schneider, Lectures on Hopf algebras)

<u>PROP 4.6</u>. Let H be a finite-dim'l Hopf algebra w/ distinguished grouplike $\alpha \in H^*$. Choose $\lambda^L \in \int^L_{H^*}$ and $\Lambda^R \in \int^R_H$ s.t. $\lambda^L(\Lambda^R) = 1$. Then $(\lambda^L, S(\Lambda^R_1), \Lambda^R_2)$ is a Frobenius system for H w/ associated Nakayama automorphism

$$N^L(h) = \alpha^{-1} \rightharpoonup S^2(h) \text{ for all } h \in H.$$

<u>PROP 4.7</u> Let H, α, g be as above. Choose $\lambda^R \in \int^R_{H^*}$ and $\Lambda^R \in \int^R_H$ s.t. $\lambda^R(\Lambda^R) = 1$, then

- $(\lambda^R, S^{-1}(\Lambda^R_2), \Lambda^R_1)$ is a Frobenius system for H w/ the corresponding Nakayama automorphism $N(h) = S^{-2}(h) \leftharpoonup \alpha^{-1}$ for all $h \in H$.
- Another Frobenius system for H is $(\lambda^R, (S\Lambda^R_1)g^{-1}, \Lambda^R_2)$ w/ the associated Nakayama automorphism $N(h) = g^{-1}(\alpha^{-1} \rightharpoonup S^2(h))\, g$.

<u>RMK</u>. As noted above, the two automorphisms N above are equal.

The first Frobenius system above is the one in PROP 4.6 for H^{cop}.

Comparing the two Nakayama automorphisms above, we obtain a shorter proof of the following crucial result of Radford.

THM 4.8. Let H, α, g be as above. Then for all $h \in H$, we have
$$S^4(h) = g(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})g^{-1}.$$

Pf. Since g and α are grouplike, so $S^2(g) = g$, $(S^*)^2(\alpha) = \alpha$. So for any $f \in H^*$, and $h \in H$, we have
$$\langle f, S^2(\alpha \rightharpoonup h)\rangle = \langle (S^*)^2(f), \alpha \rightharpoonup h\rangle = \langle (S^*)^2(f) \cdot \overset{\overset{(S^*)^2(\alpha)}{\parallel}}{\alpha}, h\rangle$$
$$= \langle (S^*)^2(f\alpha), h\rangle = \langle f\alpha, S^2(h)\rangle = \langle f, \alpha \rightharpoonup S^2(h)\rangle.$$

i.e., S^2 commutes w/ $\alpha \rightharpoonup$. Similarly, S^2 commutes w/ $\leftharpoonup \alpha$. Moreover,
$$\alpha \rightharpoonup (gxg^{-1}) = \sum \alpha\left((gxg^{-1})_2\right) \cdot (gxg^{-1})_1 = \sum \alpha(g x_2 g^{-1}) \cdot g x_1 g^{-1}$$
$$= \sum \underbrace{\alpha(g)\alpha(g^{-1})}_{\overset{\parallel}{1}} \alpha(x_2)\, g x_1 g^{-1} = g\left(\sum \alpha(x_2)x_1\right)g^{-1} = g(\alpha \rightharpoonup x)g^{-1}.$$

for all $x \in H$. By Prop 4.7, $S^{-2}(h) \leftharpoonup \alpha^{-1} = g^{-1}(\alpha^{-1} \rightharpoonup S^2(h))g$

Applying S^2 and conjugating by g, we see that

$$g\left[S^2(S^{-2}(h) \leftharpoonup \alpha^{-1})\right]g^{-1} = g\left[h \leftharpoonup \alpha^{-1}\right]g^{-1} \gg$$
$$g\left[S^2(g^{-1}(\alpha^{-1} \rightharpoonup S^2(h))g)\right]g^{-1} = \alpha^{-1} \rightharpoonup S^4(h)$$

So $g(h \leftharpoonup \alpha^{-1})g^{-1} = \alpha^{-1} \rightharpoonup S^4(h)$. Finally, apply $\alpha \rightharpoonup$ on both sides, we have $S^4(h) = g(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1})g^{-1}$. ▨

Recall by Cor. 2.6, if H and H^* are both semisimple, then $g = 1_H$ and $\alpha = \varepsilon$. In this case, $S^4 = id$. Similar results hold for weak Hopf algebras (or fusion categories), and they can be used to prove properties of global dimensions of such categories.

To show in the case when H and H^* are semisimple, then $S^2 = id$, it suffices to show that -1 cannot be an eigenvalue of S^2, and we need some facts about traces.

Recall that for any finite-dim'l vector space V, we have $V^* \otimes V \cong \mathrm{End}_{\Bbbk}(V)$. via $(\varphi \otimes v)(w) := \varphi(w) v$. Under this isom, the linear trace map. $\mathrm{Tr}_V : \mathrm{End}_{\Bbbk}(V) \to \Bbbk$ can be explicitly written as $\mathrm{Tr}_V(\varphi \otimes v) = \varphi(v)$.

<u>LEM 4.9</u>. Let A be a finite dim'l Frobenius algebra w/ Frobenius system (f, r_i, l_i). Let $e \in A$ be such that $e^2 = ce$ for some $c \in \Bbbk$. Then for any $F \in \mathrm{End}_{\Bbbk}(eA)$, we have

$$c \cdot \mathrm{Tr}_{eA}(F) = \sum_i f\Big(F(el_i)\, r_i\Big)$$

<u>Pf</u>. For any $x \in A$, $ex = \sum_i f(exr_i)\, l_i$ by def. Thus,

$$e^2 x = \sum_i f(exr_i)\, el_i, \text{ and so } cF(ex) = F(cex) = F(e^2 x)$$

$= \sum_i f(exr_i)\, F(el_i)$. Using the isom $\mathrm{End}_{\Bbbk}(V) \cong V^* \otimes V$ above for $V = eA$,

we have cF corresponds to $\sum_i f(\cdot\, r_i) \otimes F(el_i)$, and so

$$c\, \mathrm{Tr}_{eA}(F) = \sum_i f(F(el_i)\, r_i) \qquad \blacksquare$$